Analysis Meets Topology: Gauss Bonnet Theorem

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`http://www.math.utah.edu/~treiberg/AnalTopSlides.pdf`

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4. Outline.

- One-Forms, Exterior Derivative, Wedge product
- Riemannian Metric, Connection forms
- Geodesic Curvature, Geodesics
- Gaussian Curvature and its Independence of Coordinates
  - Sphere Example
- Local Gauss Bonnet Theorem
  - Proof Using Green's Theorem
  - Domains with Corners
  - Geodesic Triangles in $K < 0$, $K = 0$, $K > 0$
- Riemannian Surfaces
  - Euler Characteristic
- Global Gauss Bonnet Theorem
- Applications

Intrinsic Geometry deals with geometry that can be deduced using just measurements on the surface, such as the angle between two vectors, the length of a vector, the length of a curve and the area of a region. Since there are many ways to measure length on a surface besides using the Pythagorean Theorem as in Euclidean geometry, this leads to many Non-Euclidean geometries.

Using additional information coming from how a surface sits in space is called Extrinsic Geometry. For example a piece of paper may be spread out flat or rolled up in three space. Both have the same intrinsic properties, but differ how they're embedded.

Let us assume our surface is described by coordinates in an open set $(u, v) \in \Omega \subset \mathbf{R}^2$. At each point $(u, v) \in \Omega$ we are given an inner product which measures vectors at that point, the Riemannian Metric,

$$ds^2 = \langle \bullet, \bullet \rangle = E(u, v)\, du^2 + 2F(u, v)\, du\, dv + G(u, v)\, dv^2$$

The matrix of functions $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$ is assumed to be positive definite quadratic form.

For example if we are given vector fields (a, b, c, d depend on (x, y))

$$V = a\frac{d}{du} + b\frac{d}{dv}, \qquad W = c\frac{d}{du} + d\frac{d}{dv}$$

then their inner product at (u, v) is

$$\langle V, W \rangle = Eac + F(ad + bc) + Gbd$$

Thus for

$$V = a\frac{d}{du} + b\frac{d}{dv}, \qquad W = c\frac{d}{du} + d\frac{d}{dv}$$

the length is

$$\|V\| = \sqrt{Ea^2 + 2Fab + Gb^2}$$

and the cosine of the angle α between nonzero V and W is the usual

$$\cos\alpha = \frac{\langle V, W\rangle}{\|V\|\,\|W\|}.$$

It is convenient to express calculus using differential forms. A one-form can be integrated along a curve in Ω. It is written

$$\theta = p\,du + q\,dv$$

where p and q are functions on Ω. One-forms are dual to vector fields: one-forms may be evaluated on vector fields to yield a function. If

$$V = a\frac{d}{du} = b\frac{d}{dv}$$

then

$$\theta(v) = ap + bq.$$

If

$$\gamma : [a, b] \to \Omega$$

is a piecewise $\mathcal{C}^1$ curve, $\gamma(t) = (u(t), v(t))$, then the usual line integral is

$$\int_\gamma \theta = \int_a^b \{p[u(t), v(t)]\, u'(t) + q[u(t), v(t)]\, v'(t)\}\, dt$$

The exterior derivative of the one-form $\theta = p\,du + q\,dv$ is the two-form

$$\mathrm{d}\,\theta = \left\{ \frac{\partial q}{\partial u} - \frac{\partial p}{\partial v} \right\} du \wedge dv$$

For a piecewise $\mathcal{C}^1$ closed curve ∂D bounding a simply connected region $D \subset \Omega$, Green's Theorem (Stokes' Theorem) reads

$$\int_{\partial D} \theta = \int_D \mathrm{d}\,\theta.$$

or more familiarly

$$\int_{\partial D} p\,du + q\,dv = \int_D \left\{ \frac{\partial q}{\partial u} - \frac{\partial p}{\partial v} \right\} du\,dv$$

10. Wedge Product

A two-form may be produced by taking the wedge-product of two one-forms. If

$$\theta = p\,du + q\,dv, \qquad \eta = r\,du + s\,dv$$

then

$$\theta \wedge \eta = \{ps - qr\}\ du \wedge dv.$$

Wedge is skew symmetric

$$\theta \wedge \eta = -\eta \wedge \theta.$$

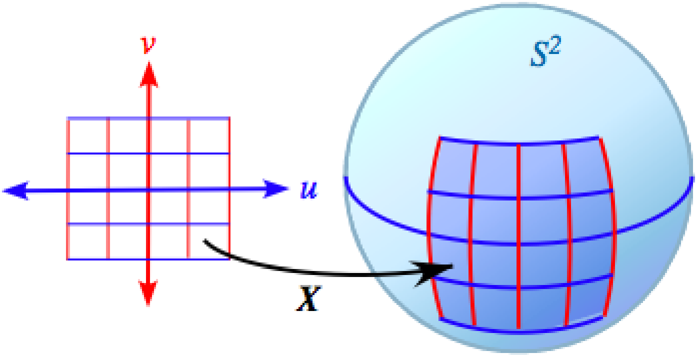
We analyze a Riemannian Surface which is given locally by a open set and Riemannian metric (Ω, ds^2). We may find vector fields V and W on Ω which are independent at all points, *e.g.*, $V = d/du$, $W = d/dv$. By applying the Gram-Schmidt Procedure, we may reduce these to ds^2-orthonormal vector fields $\mathbf{e}_1$ and $\mathbf{e}_2$ on Ω. The dual one forms θ^i are completely defined by the equations for $i, j = 1, 2$,

$$\theta^i(\mathbf{e}_j) = \delta^i{}_j = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases}$$

Then the metric may be expressed as a sum of squares of these one-forms

$$ds^2 = (\theta^1)^2 + (\theta^2)^2,$$

which amounts to the statement that a positive definite quadratic form is the sum of squares of two linear forms. If the metric can be so decomposed, then the one forms, and their corresponding dual vector fields are ds^2 orthonormal.



Consider a neighborhood of the equator of the standard unit sphere $\mathbb{S}^2 \subset \mathbf{R}^3$ given as the solution $x^2 + y^2 + z^2 = 1$. We shall consider the intrinsic metric given in latitude-longitude coordinates by pulling back the $\mathbf{R}^3$ metric. Thus $(u, v) \in (-\pi, \pi) \times (-\pi/2, \pi/2) = \Omega$ and $X : \Omega \to \mathbf{R}^3$.

$$X = \begin{pmatrix} \cos u \cos v \\ \sin u \cos v \\ \sin v \end{pmatrix}, \quad X_u = \begin{pmatrix} -\sin u \cos v \\ \cos u \cos v \\ 0 \end{pmatrix}, \quad X_v = \begin{pmatrix} -\cos u \sin v \\ -\sin u \sin v \\ \cos v \end{pmatrix}$$

so

$$E = X_u \bullet X_u = \cos^2 v, \qquad F = X_u \bullet X_v = 0, \qquad G = X_v \bullet X_v = 1.$$

The metric may be decomposed into

$$\theta^1 = \cos v \, du, \qquad \theta^2 = dv,$$
$$ds^2 = \cos^2 v \, du^2 + dv^2 = (\theta^1)^2 + (\theta^2)^2$$

Incidentally, $\mathbf{e}_1 = \sec v \, \frac{d}{du}$, $\mathbf{e}_2 = \frac{d}{dv}$.

How does the frame field change from point to point? It depends on one-form ${\omega_1}^2$, called the connection form. The connection form is uniquely determined by the equations

$$d\,\theta^1 = -\theta^2 \wedge {\omega_1}^2, \qquad d\,\theta^2 = \theta^1 \wedge {\omega_1}^2$$

For convenience we write ${\omega_2}^1 = -{\omega_1}^2$ and ${\omega_1}^1 = {\omega_2}^2 = 0$.
Then the covariant derivative or directional derivative of the frame fields are determined by

$$\nabla_{\mathbf{e}_i}\mathbf{e}_j = \sum_{k=1}^{2} {\omega_j}^k(\mathbf{e}_i)\,\mathbf{e}_k$$

which extends to general vector fields $V = \sum v^\ell \mathbf{e}_\ell$, $W = \sum w^m \mathbf{e}_m$ by linearity and Leibnitz formula

$$\nabla_V W = \sum_{\ell=1}^{2} v^\ell \sum_{m=1}^{2} \left[(\mathbf{e}_\ell w^m)\ \mathbf{e}_m + w^m \sum_{k=1}^{2} \omega_m^k(\mathbf{e}_\ell)\mathbf{e}_k \right]$$

where, as usual for a vector field $Z = a\frac{d}{dx} + b\frac{d}{dy}$ and function f,

$$Zf = af_x + bf_y.$$

15. Connection form in sphere

In the latitude/longitude coordinates, $\theta^1 = \cos v\, du$, $\theta^2 = dv$ so, evidently ${\omega_1}^2 = \sin v\, du$ since

$$\begin{aligned} -\sin v\, dv \wedge du = \mathrm{d}\,\theta^1 &= -\theta^2 \wedge {\omega_1}^2 = -dv \wedge (\sin v\, du) \\ 0 = \mathrm{d}\,\theta^2 &= \theta^1 \wedge {\omega_1}^2 = (\cos v\, du) \wedge (\sin v\, du) \end{aligned}$$

It follows that

$$\begin{aligned} \nabla_{\mathbf{e}_1}\mathbf{e}_1 &= {\omega_1}^2(\mathbf{e}_1)\,\mathbf{e}_2 = \sin v\, du\left(\sec v \frac{d}{du}\right)\frac{d}{dv} = \tan v \frac{d}{dv} \\ \nabla_{\mathbf{e}_1}\mathbf{e}_2 &= {\omega_2}^1(\mathbf{e}_1)\,\mathbf{e}_1 = -\sin v\, du\left(\sec v \frac{d}{du}\right)\sec v \frac{d}{du} = -\sec v \tan v \frac{d}{dv} \\ \nabla_{\mathbf{e}_2}\mathbf{e}_1 &= {\omega_1}^2(\mathbf{e}_2)\,\mathbf{e}_2 = \sin v\, du\left(\frac{d}{dv}\right)\frac{d}{dv} = 0 \\ \nabla_{\mathbf{e}_2}\mathbf{e}_2 &= {\omega_2}^1(\mathbf{e}_2)\,\mathbf{e}_1 = -\sin v\, du\left(\frac{d}{dv}\right)\sec v \frac{d}{du} = 0. \end{aligned}$$

The covariant derivative of $\mathbf{e}_j$ measures the rate at which the $\mathbf{e}_j$ vector field turns as we move in the $\mathbf{e}_i$ direction. Thus if $\gamma(s)$ is a unit speed curve in Ω then the geodesic curvature is the rate of turning of the tangent vector γ' in a perpendicular $(\gamma')^\perp$ direction

$$\kappa_g = \langle \nabla_{\gamma'}\gamma', (\gamma')^\perp \rangle$$

The geodesic curvature tells how much the steering wheel is turned from center as you drive along an surface embedded in three space. A curve that does not turn, that is, whose direction stays forward as it moves along the curve is called a geodesic or auto-parallel curve. For example, if you drive on a surface and keep the wheel centered, your car will follow a geodesic along the surface. Similarly, Scotch Tape, which cannot turn sideways, follows a geodesic as it is spread along a surface.

Using the latitude/longitude coordinates, we may consider first the longitude curves $\gamma(t) = (u_0, t)$, where u_0 is constant. $\gamma' = \frac{d}{dv} = \mathbf{e}_2$ so $(\gamma')^\perp = \mathbf{e}_1$. The longitude curves are geodesics because

$$\nabla_{\gamma'}\gamma' = \nabla_{\mathbf{e}_2}\mathbf{e}_2 = 0$$

On the other hand, the geodesic curvature of the latitude curves $\gamma(t) = (t, v_0)$ have

$$\gamma'(t) = \frac{d}{du} = \sin v_0\, \mathbf{e}_1.$$

Scaling to unit speed gives tangent $\frac{\gamma'}{\|\gamma_1\|} = \mathbf{e}_1$ so that the geodesic curvature is

$$\kappa_g = \langle \nabla_{\mathbf{e}_1}\mathbf{e}_1, \mathbf{e}_2 \rangle = \langle \sin v_0\, \mathbf{e}_2, \mathbf{e}_2 \rangle = \sin v_0$$

Thus longitude lines above the equator are turning toward the north pole whereas the equator $v_0 = 0$ is a geodesic.

The first differentiation of the metric frame gives the connection form, which tells how to differentiate vector fields. The second differentiation is called the Gaussian Curvature, which is a function defined over Ω.
The Gaussian Curvature K is defined by the equation

$$\mathrm{d}\,{\omega_1}^2 = -K\,\theta^1 \wedge \theta^2$$

1. In case of the plane in rectangular coordinates, ${\omega_1}^2 = 0$ so that $K = 0$ everywhere in Ω.
2. In the case of the unit sphere in latitude/longitude coordinates,

   $$\mathrm{d}\,{\omega_1}^2 = \mathrm{d}(\sin v\, du) = \cos v\, dv \wedge du = -(\cos v\, du) \wedge (dv) = -\theta^1 \wedge \theta^2$$

   so $K(x) = 1$ at all points of Ω.

Theorem (Gauss's *Theorema Egregium*, 1826)

Gauss Curvature is an invariant of the Riemannan metric on Ω.

No matter which choices of coordinates or frame fields are used to compute it, the Gaussian Curvature is the same function.
Let us suppose that $\widetilde{\mathbf{e}_1}$ and $\widetilde{\mathbf{e}_2}$ is another orthonormal frame field computed in another coordinate system $(\tilde{u}, \tilde{v})$. Pulling back these orthonormal vectors to the original coordinate system yields another orthogonal frame. Thus there is a function $\varphi(u, v)$ such that

$$\widetilde{\mathbf{e}_i} = \sum_{j=1}^{2} a_i{}^j \, \mathbf{e}_j \qquad \text{where} \qquad (a_i{}^j) = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$$

Noting that $A = (a_i{}^j)$ is skew symmetric, its inverse is its transpose $B = A^{-1} = A^T$. Hence the dual coframe for the tilde frame is given by

$$\widetilde{\theta^i} = \sum \theta^k \, b_k{}^i = \sum \theta^k \, a_i{}^k$$

Computing the connection form for the tilde frame,

$$
\begin{aligned}
\mathrm{d}\,\widetilde{\theta}^i &= \sum_{k=1}^{2} \left\{ \mathrm{d}\,\theta^k\, a_i{}^k + \theta^k \wedge \mathrm{d}\, a_i{}^k \right\} \\
&= \sum_{\ell=1}^{2} \theta^\ell \wedge \left\{ \sum_{k=1}^{2} \omega_\ell{}^k\, a_i{}^k + \mathrm{d}\, a_i{}^\ell \right\} \\
&= \sum_{m=1}^{2} \widetilde{\theta^m} \wedge \sum_{\ell=1}^{2} \left\{ \sum_{k=1}^{2} a_m{}^\ell \omega_\ell{}^k\, a_i{}^k + a_m{}^\ell\, \mathrm{d}\, a_i{}^\ell \right\}
\end{aligned}
$$

Thus I claim

$$
\widetilde{\omega_m{}^i} = \sum_{\ell=1}^{2} \left\{ \sum_{k=1}^{2} a_m{}^\ell \omega_\ell{}^k\, a_i{}^k + a_m{}^\ell\, \mathrm{d}\, a_i{}^\ell \right\} \qquad (1)
$$

This follows if we can show $\widetilde{\omega_m{}^i}$ is skew.

By the skewness of ${\omega_k}^\ell$ we have

$$\sum_{\ell=1}^{2}\sum_{k=1}^{2} {a_m}^\ell {\omega_\ell}^k\, {a_i}^k = -\sum_{\ell=1}^{2}\sum_{k=1}^{2} {a_m}^\ell {\omega_k}^\ell\, {a_i}^k = -\sum_{k=1}^{2}\sum_{\ell=1}^{2} {a_i}^k {\omega_k}^\ell\, {a_m}^\ell.$$

so the first term of (1) is skew. By differentiating

$${\delta_\ell}^m = \sum_{j=1}^{2} {a_\ell}^j\, {b_j}^m = \sum_{j=1}^{2} {a_\ell}^j\, {a_m}^j \tag{2}$$

we find

$$0 = \sum_{j=1}^{2} {a_m}^j\, \mathrm{d}\, {a_\ell}^j + \sum_{j=1}^{2} {a_\ell}^j\, \mathrm{d}\, {a_m}^j$$

so the second term of (1) is skew also.

Now we compute the tilde Gaussian Curvature.

$$\begin{aligned}
\mathrm{d}\,\widetilde{\omega_m{}^i} &= \sum_{\ell=1}^{2}\sum_{k=1}^{2}\left(a_i{}^k\,\mathrm{d}\,a_m{}^\ell \wedge \omega_\ell{}^k + a_m{}^\ell\,\mathrm{d}\,a_i{}^k \wedge \omega_\ell{}^k + a_m{}^\ell a_i{}^k\,\mathrm{d}\,\omega_1{}^2\right) \\
&\qquad + \sum_{\ell=1}^{2}\mathrm{d}\,a_m{}^\ell \wedge \mathrm{d}\,a_i{}^\ell \\
\mathrm{d}\,\widetilde{\omega_1{}^2} &= \left(a_2{}^2\,\mathrm{d}\,a_1{}^1 - a_1{}^2\,\mathrm{d}\,a_2{}^1 + a_2{}^1\,\mathrm{d}\,a_1{}^2 - a_1{}^1\,\mathrm{d}\,a_2{}^2\right)\omega_1{}^2 \\
&\quad - \left(a_1{}^1 a_2{}^2 - a_1{}^2 a_2{}^1\right) K\,\theta^1 \wedge \theta^2 + \sum_{\ell=1}^{2}\mathrm{d}\,a_1{}^\ell \wedge \mathrm{d}\,a_2{}^\ell \\
&= -K\widetilde{\theta^1} \wedge \widetilde{\theta^2}
\end{aligned}$$

so $K = \tilde{K}$. This is because the first term cancels;

the second equals

$$
\begin{aligned}
-\left({a_1}^1{a_2}^2 - {a_1}^2{a_2}^1\right) K\,\theta^1\wedge\theta^2 &= -K\left({a_1}^1\theta^1 + {a_1}^2\theta^2\right)\wedge\left({a_2}^1\theta^1 + {a_2}^2\theta^2\right)\\
&= -K\widetilde{\theta^1}\wedge\widetilde{\theta^2}.
\end{aligned}
$$

Using ${\delta_p}^q = \sum_{j=1}^2 {b_p}^j {a_j}^p = \sum_{j=1}^2 {a_j}^p {a_j}^q$, the third equals

$$
\begin{aligned}
\sum_{\ell=1}^2 \mathrm{d}\,{a_1}^\ell\wedge\mathrm{d}\,{a_2}^\ell &= \sum_{p=1}^2\sum_{q=1}^2 {\delta_p}^q\,\mathrm{d}\,{a_1}^p\wedge\mathrm{d}\,{a_2}^q\\
&= \sum_{j=1}^2\sum_{p=1}^2\sum_{q=1}^2 {a_j}^p{a_j}^q\,\mathrm{d}\,{a_1}^p\wedge\mathrm{d}\,{a_2}^q\\
&= \sum_{j=1}^2\left(\sum_{p=1}^2 {a_j}^p\,\mathrm{d}\,{a_1}^p\right)\wedge\left(\sum_{q=1}^2 {a_j}^q\,\mathrm{d}\,{a_2}^q\right) = 0
\end{aligned}
$$

because each parenthesis is skew the first is zero if $j = 1$ and the second is zero if $j = 2$. □

Theorem (Gauss Bonnet)

Let $D \subset \Omega$ be a region bounded by a continuously differentiable simple closed curve ∂D. Then

$$\int_D K\,dA + \int_{\partial D} \kappa_g\,ds = 2\pi$$

where K is the Gaussian Curvature function, dA is the area element and κ_g is the geodesic curvature of the boundary curve ∂D.

For example, in the flat plane $K = 0$, then the integral of geodesic curvature is just the total angle around the closed curve thus

$$\int_D K\,dA + \int_{\partial D} \kappa_g\,ds = 0 + 2\pi.$$

For example, on the unit sphere $\mathbb{S}^2$, let D be the upper hemisphere. Then since $K = 1$ on D and that ∂D is a great circle thus geodesic $\kappa_g = 0$, the left side of the equation is

$$\int_D K\,dA + \int_{\partial D} \kappa_g\,ds = \mathrm{A}(D) + 0 = 2\pi.$$

The area form is $dA = \theta^1 \wedge \theta^2$. Applying Green's Theorem to the connection form, we find

$$\int_D K\, dA = -\int_D \mathrm{d}\,{\omega_1}^2 = -\int_{\partial D} {\omega_1}^2.$$

Let $\gamma(s)$ be a unit speed positively oriented parameterization of the curve ∂D (D is to the left as you follow ∂D in the V direction). The unit tangent vector is

$$\gamma' = V = \cos\varphi\, \mathbf{e}_1 + \sin\varphi\, \mathbf{e}_2$$

where $\varphi(s)$ is the angle between $\mathbf{e}_1$ and $V = \gamma'$. Let

$$V^\perp = -\sin\varphi\, \mathbf{e}_1 + \cos\varphi\, \mathbf{e}_2$$

Computing the covariant derivative of V along ∂D,

$$\begin{aligned}
\nabla_V V &= \nabla_V \left(\cos\varphi\, \mathbf{e}_1 + \sin\varphi\, \mathbf{e}_2\right) \\
&= \left(-\sin\varphi\, \mathbf{e}_1 + \cos\varphi\, \mathbf{e}_2\right) \frac{d\varphi}{ds} + \\
&\quad + \cos\varphi \left(\cos\varphi \nabla_{\mathbf{e}_1} + \sin\varphi \nabla_{\mathbf{e}_2}\right) \mathbf{e}_1 + \sin\varphi \left(\cos\varphi \nabla_{\mathbf{e}_1} + \sin\varphi \nabla_{\mathbf{e}_2}\right) \mathbf{e}_2 \\
&= \frac{d\varphi}{ds} V^\perp + \cos^2\varphi \nabla_{\mathbf{e}_1}\mathbf{e}_1 + \cos\varphi\, \sin\varphi \nabla_{\mathbf{e}_2}\mathbf{e}_1 \\
&\quad + \sin\varphi\, \cos\varphi \nabla_{\mathbf{e}_1}\mathbf{e}_2 + \sin^2\varphi \nabla_{\mathbf{e}_2}\mathbf{e}_2 \\
&= \frac{d\varphi}{ds} V^\perp + \cos^2\varphi\, {\omega_1}^2(\mathbf{e}_1)\mathbf{e}_2 + \cos\varphi \sin\varphi\, {\omega_1}^2(\mathbf{e}_2)\mathbf{e}_2 \\
&\quad + \sin\varphi \cos\varphi\, {\omega_2}^1(\mathbf{e}_1)\mathbf{e}_1 + \sin^2\varphi\, {\omega_2}^1(\mathbf{e}_2)\mathbf{e}_1 \\
&= \frac{d\varphi}{ds} V^\perp + \cos\varphi\, {\omega_1}^2(V)\mathbf{e}_2 + \sin\varphi\, {\omega_2}^1(V)\mathbf{e}_1 \\
&= \left(\frac{d\varphi}{ds} + {\omega_1}^2(V)\right) V^\perp
\end{aligned}$$

Then the geodesic curvature of ∂D is given by

$$\begin{aligned}\kappa_g &= \langle \nabla_V V, V^\perp \rangle \\ &= \tfrac{d\varphi}{ds} + {\omega_1}^2(V)\end{aligned}$$

Hence the integral equals

$$\begin{aligned}\int_D K\,dA &= -\int_{\partial D} {\omega_1}^2 \\ &= -\int_0^L {\omega_1}^2(V)\,ds \\ &= -\int_0^L \left(\kappa_g - \tfrac{d\varphi}{ds}\right) ds \\ &= -\int_0^L \kappa_g\,ds + 2\pi\end{aligned}$$

since the total change in angle relative to the nonvanishing vector field $\mathbf{e}_1$ in Ω going once around in the positive direction equals $\varphi(L) - \varphi(0) = 2\pi$. $\square$

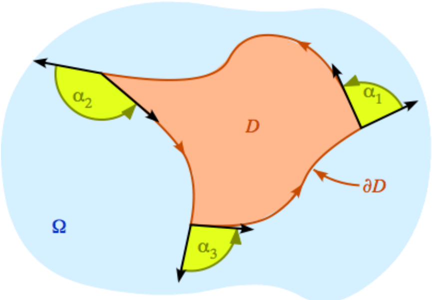
28. Local Gauss Bonnet for Domains with Corners

Theorem (Gauss Bonnet)

Let $D \subset \Omega$ be a region bounded by a piecewise $\mathcal{C}^1$ simple closed curve $\gamma = \partial D$ with corners at the points $\gamma(s_i)$, $i = 1, \ldots, n$. Then

$$\int_D K\, dA + \int_{\partial D} \kappa_g\, ds + \sum_{i=1}^n \alpha_i = 2\pi$$

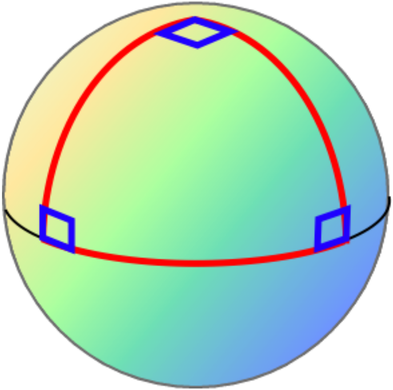
where K, dA, κ_g are the Gaussian Curvature, area form and geodesic curvature, as before, and α_i is the exterior angle, the angle from $\gamma'(s_i-)$ to $\gamma'(s_i+)$ at the corner $\gamma(s_i)$.



For example, if D is a triangle in the flat plane, and $\beta_i = \pi - \alpha_i$ are the interior angles at the vertices, then the left side is just the total angle around the triangle

$$\begin{aligned} 2\pi &= \int_D K\, dA + \int_{\partial D} \kappa_g\, ds + \textstyle\sum_{i=1}^{3} \alpha_i = 0 + 0 + \alpha_1 + \alpha_2 + \alpha_3 \\ &= 3\pi - \beta_1 - \beta_2 - \beta_3, \end{aligned}$$

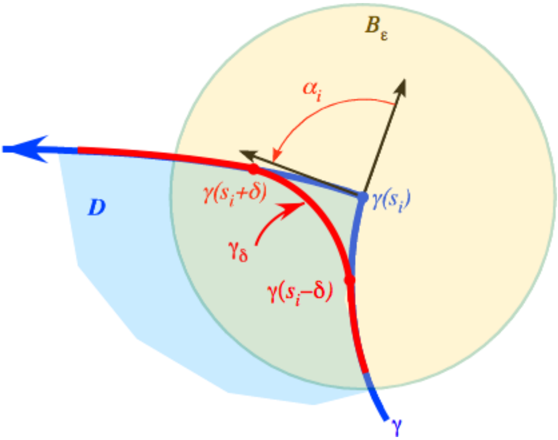
In other words the sum of the interior angles $\beta_1 + \beta_2 + \beta_3 = \pi$.



Let D be the region of the unit sphere in the first orthant with right angled corners at $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$

The sides are great circles which are geodesics, so $\kappa_g = 0$. The Gaussian Curvature is $K = 1$. Thus the Local Gauss Bonnet formula with corners yields

$$\int_D K\,dA + \int_{\partial D} \kappa_g\,ds + \sum_{i=1}^{n} \alpha_i$$
$$= \mathrm{A}(D) + 0 + \frac{3\pi}{2}$$
$$= \frac{4\pi}{8} + \frac{3\pi}{2} = 2\pi$$



The idea is to approximate γ (blue curve) with kinks at the corners $\gamma(s_i)$, $i = 1, \ldots, n$ with a nearby curve γ_δ (red curve) that equals γ away from the kinks but smoothly rounds out the corners. Applying the Local Gauss Bonnet to γ_δ and taking the limit $\delta \to 0$ gives the result.

To give more detail, suppose that the closed ϵ balls centered at the corners are all contained in Ω. Then since the union $K = \cup_{i=1}^{n} \overline{B(\gamma(s_i), \epsilon)}$ is a compact set, there is a constant $c < \infty$ such that

$$\left|{\omega_1}^2(x)[V]\right| \le c \qquad \text{for all } x \in K \text{ and unit vector fields } V$$

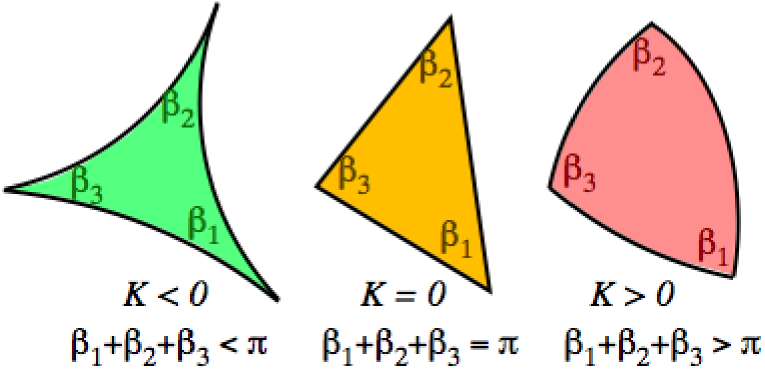
It follows that near the kinks, the integral of the approximating curve

$$\left| \int_{s_i-\delta}^{s_i+\delta} \kappa_g \, ds - \varphi(s_i + \delta) + \varphi(s_i - \delta) \right|$$

$$= \left| \int_{s_i-\delta}^{s_i+\delta} \left(\kappa_g - \frac{d\varphi}{ds} \right) ds \right| = \left| \int_{\gamma_\delta[s_i-\delta, s_i+\delta]} {\omega_1}^2 \right| \le 2c \, \mathrm{L}(\gamma_\delta[s_i - \delta, s_i + \delta])$$

which tends to zero as $\delta \to 0$. Since $\varphi(s_i + \delta) - \varphi(s_i - \delta) \to \alpha_i$, it follows that

$$2\pi = \lim_{\delta \to 0} \int_{D_\delta} K \, dA + \int_{\gamma_\delta} \kappa_g \, ds = \int_D K \, dA + \int_{\partial D} \kappa_g \, ds + \sum_{i=1}^{n} \alpha_i. \qquad \square$$



A geodesic triangle T is the region bounded by three geodesics that meet in three points. By rewriting the Local Gauss Bonnet Theorem, the integral curvature may be regarded as the correction to π of the sum of interior angles.

$$\beta_1 + \beta_2 + \beta_3 = \pi + \int_D K \, dA$$

Gauss published this theorem in case K is constant.

Karl Friedrich Gauss (1777-1855)

Gauss, who worked at Göttingen all of his life, made huge contributions to algebra, number theory, complex functions, geodesy, magnetism and optics. He wrote his opus on geometry *Disquitiones Generales circa Superficias Curvas* fairly late in his career in 1827. There he commented about his theorem about triangles

> *"this theorem, if we mistake not, ought to counted among the most elegant in the theory of curved surfaces."*

The sphere is an example of a surface that cannot be covered by one coordinate chart. The idea is to cover the surface by many charts in which computations are consistent chart to chart.

A Riemannian surface or Riemannian manifold of dimension two, S, is a topological space with a family of maps $X_\alpha : \Omega_\alpha \to S$ such that $\Omega_\alpha \subset \mathbf{R}^2$ is an open set, and a family of Riemannian metrics $g_\alpha(x)\langle \bullet, \bullet \rangle_\alpha$ on each Ω_α such that

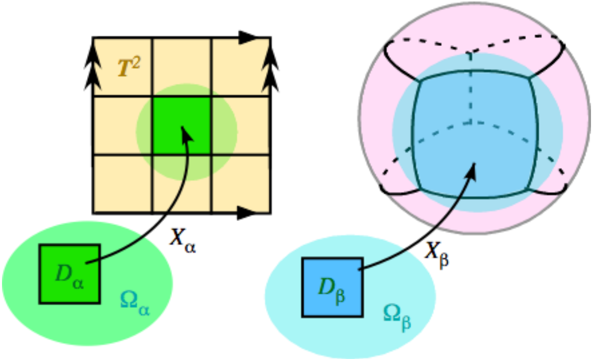
- $S = \cup_\alpha U_\alpha$
- For each pair α, β with $X_\alpha(\Omega_\alpha) \cap X_\beta(\Omega_\beta) = W \neq \emptyset$ we have $X_\alpha^{-1}(W)$, $X_\beta^{-1}(W)$ are open sets in $\mathbf{R}^2$ and $X_\alpha^{-1} \circ X_\beta$ and $X_\beta^{-1} \circ X_\alpha$ are differentiable maps. If these maps also have positive Jacobian determinant, we say the surface is orientable.
- Also metrics are consistently defined: for all α, β as above, vector fields U, V on $X_\alpha^{-1}(W) \subset \Omega_\alpha$, if $y = X_\beta^{-1} \circ X_\alpha(x)$,

$$g_\beta(y)\langle \mathrm{d}_x\, (X_\beta^{-1} \circ X_\alpha)(U), \mathrm{d}_x\, (X_\beta^{-1} \circ X_\alpha)(V) \rangle = g_\alpha(x)\langle U, V \rangle$$

The pair $(\Omega_\alpha, X_\alpha)$ is called a local parameterization or coordinate system.

We would like to extend the formula to closed surfaces such as the torus and the sphere.

A polygonal decomposition $\mathcal{P}$ of a closed surface S is a finite collection of one-to-one coordinate charts $X_\alpha : \Omega_\alpha \to S$ and corresponding regions $D_\alpha \subset \Omega_\alpha$ bounded by piecewise smooth curves, such that their images $X_\alpha(D_\alpha)$ cover S in such a way that if any two overlap, they do so in either a single common vertex or a single common edge. Every compact surface S has a polygonal decomposition.

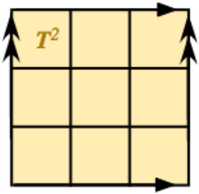


Theorem (Euler Characteristic of S)

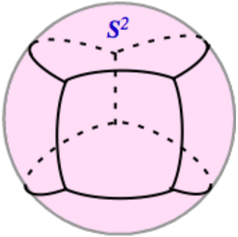
If $\mathcal{P}$ is a polygonal decomposition of the compact surface S and v, e and f are the numbers of vertices, edges and faces in $\mathcal{P}$, then the the integer

$$\chi(S) = v - e + f$$

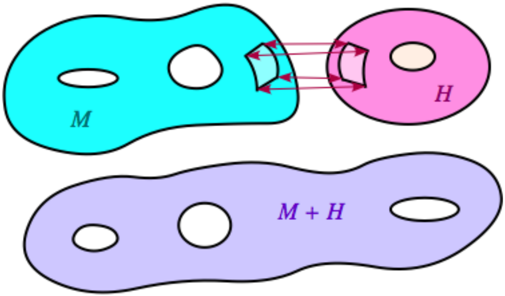
is the same for all polygonal decompositions of S.
$\chi(S)$ is called the Euler Characteristic of S.



$v=9$, $e=18$, $f=9$
$\chi(T^2) = 9\text{-}18\text{+}9 = 0$



$v=6$, $e=9$, $f=5$
$\chi(S^2) = 6\text{-}9\text{+}5 = 2$



Theorem

Adding a handle H to a compact surface M decreases the Euler Characteristic by two $\chi(M+H) = \chi(M) - 2$.

In the diagram, a four sided face is removed from both M and H and the surfaces are glued along the four edges. Thus $M + H$ has four fewer vertices, four fewer edges and two fewer faces than $M \cup H$.

Theorem

Diffeomorphic surfaces have the same Euler Characteristic.

The polygonal decomposition of one surface is mapped to a polygonal decomposition of the second with the same number of vertices, edges and faces. In fact, Euler Characteristic is preserved by homeomorphism, so is a topological invariant.

Suppose we start with a sphere Σ and successively add h handles $(h = 0, 1, 2, \ldots)$ to obtain a new surface $\Sigma(h)$. Then

$$\chi(\Sigma(h)) = \chi(\Sigma) - 2h = 2 - 2h.$$

Theorem

Every compact orientable surface is diffeomorphic to some $\Sigma(h)$.

Theorem (Gauss Bonnet Theorem)

Let S be a compact, oreintable, Riemannian surface. Then its total curvature equals 2π times its Euler Characteristic.

$$\int_S K\,dA = 2\pi\chi(S)$$

The Gauss Bonnet Theorem has many deep corollaries.

Corollary

Suppose that S is a compact oriented Riemannian surface whose Gauss Curvature is everywhere positive $K(x) > 0$ for all $x \in S$. Then S is diffeomorphic to the standard sphere.

An oriented surface must be diffeomorphic to one of the $\Sigma(h)$'s. However, the total curvature

$$0 < \frac{1}{2\pi}\int_S K\,dA = \chi(S) = \chi(\Sigma(h)) = 2 - 2h$$

implies that $h = 0$ and S is diffeomorphic to the sphere $\Sigma(0)$.

Pierre Ossian Bonnet (1819–1892)

Bonnet made many important contributions to the differential geometry of surfaces. He sharpened and recognized the importance of many of Gauss's theorems

Bonnet published the local theorem for variable curvature for domains with corners in *Journ. Ecole Polytechnique* **19** (1848).
This proof using Green's Theorem was noticed by Gaston Darboux in 1894.

Fix a polygonal decomposition of $S = \cup_{i=1}^{f} X_i(D_i)$ which has v vertices, e edges and f faces. Suppose that the polygon D_i has e_i edges and thus e_i vertices. Call the interior angles $\beta_{i,j}$. Then the Local Gauss Bonnet Theorem applied to one polygon may be written

$$\int_{D_i} K\, dA = 2\pi - \sum_{j=1}^{e_i} (\pi - \beta_{i,j}) - \int_{\partial d_i} \kappa_g\, ds$$

Summing the integrals of curvature, and applying this on each polygon

$$\begin{aligned} \int_S K\, dA &= \sum_{i=1}^{f} \int_{D_i} K\, dA \\ &= \sum_{i=1}^{f} \left\{ 2\pi - \sum_{j=1}^{e_i} (\pi - \beta_{i,j}) - \int_{\partial d_i} \kappa_g\, ds \right\} \end{aligned}$$

Each edge occurs exactly twice in the sum. Since S is positively oriented then a shared edge between two neighboring polygons is integrated in opposite directions in the two polygons, resulting in cancellation of the all line integrals.

Since each edge occurs in exactly two faces

$$\sum_{i=1}^{f} e_i = 2e.$$

Because the surface is smooth at each vertex, the sum of interior angles around each vertex is 2π. Also, there are v vertices so

$$\sum_{i=1}^{f} \sum_{j=1}^{e_i} \beta_{i,j} = 2\pi v.$$

Inserting into the sum,

$$\int_S K\, dA = 2\pi f - 2\pi e + 2\pi v = 2\pi\chi(S). \qquad \square$$

Note that the same argument may be applied to any region $R \subset S$ which has a polygonal decomposition. The line integrals on the boundary of R and exterior angles don't cancel yielding

$$\int_R K\, dA + \int_{\partial R} \kappa_g\, ds + \sum_{\text{corners of } \partial R} \alpha_i = 2\pi\chi(R)$$

Here is the Poincaré Plane, a Riemannian surface whose Gaussian curvature is $K = -1$. Let $\Omega = \{(u,v) \in \mathbf{R}^2 : u^2 + v^2 < 1\}$ be the unit disk. Then the Poincaré metric is

$$ds^2 = \frac{4(du^2 + dv^2)}{(1 - u^2 - v^2)^2}.$$

An orthonormal frame is

$$\theta^1 = \frac{2\,du}{1 - u^2 - v^2}, \qquad \theta^2 = \frac{2\,dv}{1 - u^2 - v^2}.$$

Differentating we find the connection form

$$\begin{aligned}
\mathrm{d}\,\theta^1 &= \frac{4v\,dv \wedge du}{(1 - u^2 - v^2)^2} = \theta^2 \wedge {\omega_2}^1 = \frac{2\,dv}{1 - u^2 - v^2} \wedge \frac{2(v\,du - u\,dv)}{1 - u^2 - v^2}\\
\mathrm{d}\,\theta^2 &= \frac{4u\,du \wedge dv}{(1 - u^2 - v^2)^2} = \theta^1 \wedge {\omega_1}^2 = \frac{2\,du}{1 - u^2 - v^2} \wedge \frac{2(u\,dv - v\,du)}{1 - u^2 - v^2}
\end{aligned}$$

Differentiating the connection form we find the curvature

$$\begin{aligned}
\mathrm{d}\,{\omega_1}^2 &= \mathrm{d}\left(\frac{2(u\,dv - v\,du)}{1-u^2-v^2}\right) \\
&= \frac{4du\wedge dv}{1-u^2-v^2} + \frac{2(u\,du+v\,dv)\wedge 2(u\,dv - v\,du)}{\left(1-u^2-v^2\right)^2} \\
&= \frac{4\left(1-u^2-v^2\right)du\wedge dv + 4\left(u^2+v^2\right)du\wedge dv}{\left(1-u^2-v^2\right)^2} \\
&= \frac{4\,du\wedge dv}{\left(1-u^2-v^2\right)^2} = \theta^1\wedge\theta^2
\end{aligned}$$

thus $K=-1$.

In the upper halfspace $\Omega = \{(u, v) \in \mathbf{R}^2 : v > 0\}$, the metric is

$$ds^2 = \frac{du^2 + dv^2}{v^2}; \qquad \theta^1 = \frac{du}{v}; \qquad \theta^2 = \frac{dv}{v}$$

so $\mathbf{e}_1 = v \frac{d}{du}$, $\mathbf{e}_2 = v \frac{d}{dv}$. Then

$$\mathrm{d}\,\theta^1 = -\frac{dv \wedge du}{v^2} = \theta^2 \wedge {\omega_2}^1 = \frac{dv}{v} \wedge \left(-\frac{du}{v}\right)$$

$$\mathrm{d}\,\theta^2 = 0 = \theta^1 \wedge {\omega_1}^2 = \frac{dv}{v} \wedge \left(\frac{du}{v}\right)$$

Thus ${\omega_1}^2 = \frac{du}{v} = \theta^1$. Also

$$\mathrm{d}\,{\omega_1}^2 = \mathrm{d}\,\theta^1 = \theta^2 \wedge {\omega_2}^1 = -\theta^2 \wedge \theta^1 = \theta^1 \wedge \theta^2$$

so $K = -1$.
Let $\gamma(s) = (u(s), v(s)) \in \Omega$ be a geodesic. Then it satisfies the ODE

$$\nabla_{\gamma'}\gamma' = 0.$$

Writing the tangent vector

$$\gamma'(s) = u'(s)\frac{d}{du} + v'(s)\frac{d}{dv} = \frac{u'}{v}\mathbf{e}_1 + \frac{v'}{v}\mathbf{e}_2$$

Thus using ${\omega_1}^2 = \theta^1$,

$$\begin{aligned}
\nabla_{\gamma'}\gamma' &= \nabla_{\gamma'}\left(\frac{u'}{v}\mathbf{e}_1 + \frac{v'}{v}\mathbf{e}_2\right) \\
&= \left(\frac{u'}{v}\right)'\mathbf{e}_1 + \frac{u'}{v}\nabla_{\gamma'}\mathbf{e}_1 + \left(\frac{v'}{v}\right)'\mathbf{e}_2 + \frac{v'}{v}\nabla_{\gamma'}\mathbf{e}_2 \\
&= \left(\frac{u'}{v}\right)'\mathbf{e}_1 + \frac{u'}{v}\left(\frac{u'}{v}\nabla_{\mathbf{e}_1}\mathbf{e}_1 + \frac{v'}{v}\nabla_{\mathbf{e}_2}\mathbf{e}_1\right) \\
&\qquad + \left(\frac{v'}{v}\right)'\mathbf{e}_2 + \frac{v'}{v}\left(\frac{u'}{v}\nabla_{\mathbf{e}_1}\mathbf{e}_2 + \frac{v'}{v}\nabla_{\mathbf{e}_2}\mathbf{e}_2\right) \\
&= \left(\frac{u'}{v}\right)'\mathbf{e}_1 + \frac{(u')^2}{v^2}{\omega_1}^2(\mathbf{e}_1)\mathbf{e}_2 + \frac{u'v'}{v^2}{\omega_1}^2(\mathbf{e}_2)\mathbf{e}_1 \\
&\qquad + \left(\frac{v'}{v}\right)'\mathbf{e}_2 + \frac{u'v'}{v^2}{\omega_2}^1(\mathbf{e}_1)\mathbf{e}_1 + \frac{(v')^2}{v^2}\omega_2(\mathbf{e}_2)\mathbf{e}_1 \\
&= \left(\frac{u'}{v}\right)'\mathbf{e}_1 + \frac{(u')^2}{v^2}\mathbf{e}_2 + \left(\frac{v'}{v}\right)'\mathbf{e}_2 - \frac{u'v'}{v^2}\mathbf{e}_1
\end{aligned}$$

Grouping the equations by basis, the equation for a geodesic is

$$\left(\frac{u'}{v}\right)' - \frac{u'v'}{v^2} = 0$$
$$\left(\frac{v'}{v}\right)' + \frac{(u')^2}{v^2} = 0$$

If $u' = 0$ then the first equation says $u' = 0$ for all s. Hence vertical curves $\gamma(s) = (u_0, ae^{bs})$ are geodesics.
The other geodesics may be written down explicitly. Indeed the curves

$$u(s) = k + a\tanh[b(s - s_0)], \qquad v(s) = a\,\mathrm{sech}[b(s - s_0)]$$

depend on four constants and satisfy the ODE. Checking,

$$u' = ab\,\mathrm{sech}^2[b(s - s_0)], \quad v' = -ab\tanh[b(s - s_0)]\,\mathrm{sech}[b(s - s_0)],$$
$$\frac{u'}{v} = b\,\mathrm{sech}[b(s - s_0)], \qquad \frac{v'}{v} = -b\tanh[b(s - s_0)]$$

Hence the ODE's are stisfied.

$$\left(\frac{u'}{v}\right)' = b^2 \operatorname{sech}[b(s-s_0)] \tanh[b(s-s_0)] = \frac{u'v'}{v^2}$$
$$\left(\frac{v'}{v}\right)' = -b^2 \operatorname{sech}^2[b(s-s_0)] = -\frac{(u')^2}{v^2}$$

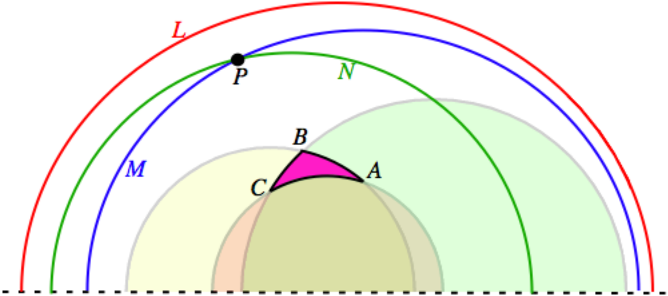
When $a > 0$ and $b \neq 0$ then the curves

$$u(s) = k + a \tanh[b(s-s_0)], \qquad v(s) = a \operatorname{sech}[b(s-s_0)]$$

are semicircles of radius a centered on $(k, 0)$ since

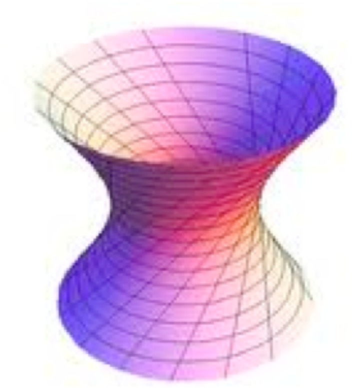
$$(u-k)^2 + v^2 = a^2 \tanh^2[b(s-s_0)] + a^2 \operatorname{sech}^2[b(s-s_0)] = a^2.$$

50. Geodesics of the Upper Half-plane Model - -



Geodesics in the Upper Half-plane Model are semicircles centered on the x-axis. The sum of the interior angles of triangle ABC is less than π.

We can see this is a Non-Euclidean geometry. It violates the parallel postulate. Indeed, through the point P there are at least two geodesic lines, M and N, that are parallel to the given line L not containing P. (L and M are parallel because they do not intersect in the Upper Half-plane.)



The hyperboloid of one sheet H is the surface detrmined by

$$x^2 + y^2 = 1 + z^2$$

Corollary

The only simple closed geodesic on the hyperboloid of one sheet is the equatorial circle $z = 0$.

We'll show that the Gauss Curvature is negative. Simple means that the geodesic curve γ has no self-intersections. Either γ meets $z = 0$ or not. If not, then it either winds around the hyperboloid or not. If not then it encloses a domain $D \subset H$ on which the Local Gauss Bonnet formula says

$$2\pi = \int_D K\,dA + \int_\gamma \kappa_g\,ds < 0 + 0$$

which is a contradiction.

Similarly, if γ loops araound H then there is an annular region U bounded by γ and $z = 0$. Then, the corresponding global Gauss Bonnet Theorem says

$$0 = 2\pi\chi(U) = \int_u K\, dA + \int_\gamma \kappa_g\, ds < 0 + 0$$

which is also a contradiction.
Finally, if γ meets $z = 0$ then it must cross it transversally, otherwise the curves agree. Then there must be at least two crossing points since γ must recross to close up. Then consider the geodesic bigon B bounded by segments of γ and $z = 0$ between two consecutive crossing points. Since the γ part of ∂B is entirely above or below $z = 0$, we must have the exterior angles $0 < \alpha_i < \pi$ at the two corners. The Local Gauss Bonnet Theorem with corners says

$$2\pi = \int_B K\, dA + \int_{\partial B} \kappa_g\, ds + \alpha_1 + \alpha_2 < 0 + 0 + \pi + \pi$$

which is also a contradiction.

It remains to show $K < 0$ for the hyperboloid. Let $f(v) = \sqrt{1+v^2}$, then the parameterization of H may be given by

$$X = \begin{pmatrix} f(v)\cos u \\ f(v)\sin u \\ v \end{pmatrix}, \quad X_u = \begin{pmatrix} -f(v)\sin u \\ f(v)\cos u \\ 0 \end{pmatrix}, \quad X_v = \begin{pmatrix} f'(v)\cos u \\ f'(v)\sin u \\ 1 \end{pmatrix},$$

so

$$E = X_u \bullet X_u = f(v)^2, \qquad F = X_u \bullet X_v = 0, \qquad G = X_v \bullet X_v = 1+f'(v)^2.$$

The metric may be decomposed into

$$\theta^1 = f(v)\,du, \qquad \theta^2 = \sqrt{1+f'(v)^2}\,dv$$
$$ds^2 = f(v)^2\,du^2 + (1+f'(v)^2)\,dv^2 = (\theta^1)^2 + (\theta^2)^2$$

Incidentally, $\mathbf{e}_1 = \frac{1}{f(v)}\frac{d}{du}$, $\mathbf{e}_2 = \frac{1}{\sqrt{1+f'(v)^2}}\frac{d}{dv}$.

Computing the connection form

$$f'(v)\,dv \wedge du = \mathrm{d}\,\theta^1 = \theta^2 \wedge {\omega_2}^1 = \sqrt{1+f'(v)^2}\,dv \wedge \left(\frac{f'(v)\,du}{\sqrt{1+f'(v)^2}}\right)$$

$$0 = \mathrm{d}\,\theta^2 = \theta^1 \wedge {\omega_1}^2 = (f(v)\,du) \wedge \left(-\frac{f'(v)\,du}{\sqrt{1+f'(v)^2}}\right)$$

It follows that

$$\mathrm{d}\,{\omega_2}^1 = \mathrm{d}\left(\frac{f'(v)\,du}{\sqrt{1+f'(v)^2}}\right) = \frac{f''(v)\,dv \wedge du}{\left[1+f'(v)^2\right]^{3/2}} = K(x)\,\theta^1 \wedge \theta^2$$

so that

$$K = -\frac{f''(v)}{f(v)\left[1+f'(v)^2\right]^2}.$$

In case $f(v) = \sqrt{1+v^2}$ we have $f''(v) = (1+v^2)^{-3/2}$ so $K < 0$.

Theorem

Let S be a compact, positively oriented surface with positive gaussian Curvature. Then any two simple closed geodesics intersect.

We have already noticed that S is diffeomorphic to the sphere. Suppose two simple closed geodesics $gamma_1$ and γ_2 don't intersect. Then the set between the two curves is a region that has both curves as boundary $\partial R = \gamma_1 \cup \gamma_2$ and that has the topology of a cylinder so $\chi(R) = 0$. The Gauss Bonnet Theorem applies to this region to yield

$$0 = 2\pi\chi(R) = \int_R K\, dA + \int_{\partial R} \kappa_g\, ds > 0 + 0$$

which is a contradiction. □

Thanks!