1. Solve the following initial value problem for the unknown function \( y(x) \).

\[
y' = 2xy^2 \quad y(1) = \frac{1}{2}
\]

SOLUTION:

Use separation of variables:

\[
\int y^{-2} \, dy = \int 2x \, dx
\]

\[-y^{-1} = x^2 + C\]

\[y(x) = \frac{-1}{x^2 + C}\]

Then use the initial condition to solve for \( C \):

\[y(1) = \frac{-1}{1 + C} = \frac{1}{2}\]

so \( C = -3 \), and our final solution is

\[y(x) = \frac{1}{3 - x^2}\]
2. Find a general solution \( y(x) \) to the following differential equation.

\[
y' = \frac{y}{x} + 2x^2
\]

**SOLUTION:**

The equation is not separable, but it is linear, so we put the equation in the following form:

\[
y' - x^{-1}y = 2x^2
\]

and use the integrating factor

\[
e^{-\frac{1}{x}} \int dx = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}
\]

Multiply through by the integrating factor and proceed:

\[
x^{-1}y' - x^{-2}y = 2x
\]

\[
\frac{d}{dx} \left( x^{-1}y \right) = 2x
\]

\[
x^{-1}y = \int 2x \, dx
\]

\[
x^{-1}y = x^2 + C
\]

\[
y(x) = x^3 + Cx
\]
3. Given: the function

\[ y(x) = c_1 e^x + c_2 e^x \cos x \]

is a solution to the differential equation

\[ y''' - 3y'' + 4y' - 2y = 0 \]

for any pair of constants \((c_1, c_2)\).

(a) Show that there is no pair of constants \((c_1, c_2)\) for which \(y(x)\) above satisfies the following initial conditions.

\[ y(0) = 1 \quad y'(0) = 0 \quad y''(0) = 0 \]

SOLUTION:

\[ y(x) = c_1 e^x + c_2 e^x \cos x \]

\[ y(0) = c_1 + c_2 = 1 \]

\[ y'(x) = c_1 e^x + c_2 (e^x \cos x - e^x \sin x) \]

\[ y'(0) = c_1 + c_2 = 0 \]

The equations \(c_1 + c_2 = 1\) and \(c_1 + c_2 = 0\) are inconsistent, so there is no solution \((c_1, c_2)\).

(b) Do the functions \(y_1 = e^x\) and \(y_2 = e^x \cos x\) form a basis for the solution space of the differential equation? Why or why not?

SOLUTION:

No they do not form a basis. The easiest argument to make is that the differential equation is linear, homogeneous, and third order, so it requires three linearly independent solutions to form a basis for the solution space.
4. A horizontal spring is attached to a stationary wall on one end and to a 3 kg mass on the other end. The spring constant for the spring is 75 N/m, and there is no damping. At time zero, the spring is at its equilibrium position and the mass is pushed towards the wall with an initial velocity of 10 m/s. The mass is then allowed to move freely. Answer the following two questions:

- What is the maximum distance from its equilibrium position that the mass will reach?
- At what time does the mass first reach its maximum distance from the wall? You may leave your answer in terms of $\pi$.

**SOLUTION:**

The full spring equation describing the motion of the mass is

$$mx'' + cx' + kx = f(t)$$

where $x(t)$ is the distance of the mass from its equilibrium position at time $t$ ($x = 0$ at equilibrium, $x > 0$ away further from the wall — the direction the spring stretches). In this case we have $m = 3$ (mass), $c = 0$ (no damping), $k = 75$ (spring constant), and $f(t) = 0$ (allowed to move freely after time zero). We also need to express the initial conditions, which are $x(0) = 0$ and $x'(0) = -10$.

Thus we have the homogeneous, constant-coefficient equation

$$3x'' + 75x = 0$$

The characteristic equation is $3r^2 + 75 = 0$, which has the imaginary solutions $r = \pm 5i$. Therefore the general solution to the differential equation is

$$x(t) = c_1 \sin 5t + c_2 \cos 5t$$

Applying the initial conditions we get

$$x(t) = -2 \sin 5t$$

The farthest the mass will reach from equilibrium is the amplitude of the function, or 2 meters. The first time the mass reaches its farthest
distance from the wall will be lowest value of $t$ that makes $x(t) = 2$, so we solve:

$$-2 \sin 5t = 2$$

$$\sin 5t = -1$$

$$5t = \frac{3\pi}{2}$$

$$t = \frac{3\pi}{10}$$
5. Find a general solution $y(x)$ to the following differential equation using the method of undetermined coefficients.

$$ y'' - 2y' + y = e^x $$

First we solve for the complementary solution $y_c$, which is the general solution to the associated homogeneous equation $y'' - 2y' + y = 0$. The characteristic equation $r^2 - 2r + 1 = 0$ has the repeated root $r = 1$. Therefore the complementary solution is

$$ y_c(x) = c_1 e^x + c_2 xe^x $$

Using undetermined coefficients to find a particular solution $y_p$ to the full equation, we would normally assume the form $y_p(x) = Ae^x$. However, this would be a duplication of a complementary solution. Multiplying by $x$ and using $y_p = Axe^x$ would also be a duplication. Multiplying by $x^2$ avoids duplication, so we try

$$ y_p = Ax^2 e^x, \quad y_p' = Ax^2 e^x + 2Axe^x, \quad y_p'' = Ax^2 e^x + 4Axe^x + 2Ae^x $$

Plug these in and solve for $A$:

$$ (Ax^2 e^x + 4Axe^x + 2Ae^x) - 2(Ax^2 e^x + 2Axe^x) + Ax^2 e^x = e^x $$

$$ (A - 2A + A)x^2 e^x + (4A - 4A)xe^x + 2Ae^x = e^x $$

$$ 2Ae^x = e^x $$

$$ A = \frac{1}{2} $$

So our overall general solution is

$$ y(x) = y_c(x) + y_p(x) $$

$$ y(x) = c_1 e^x + c_2 xe^x + \frac{1}{2} x^2 e^x $$
6. Find a general solution $y(x)$ to the following differential equation using the method of variation of parameters.

$$y'' - 2y' + 2y = e^x \csc x$$

**SOLUTION:**

First we solve for the complementary solution $y_c$, which is the general solution to the associated homogeneous equation $y'' - 2y' + 2y = 0$. The characteristic equation $r^2 - 2r + 2 = 0$ has the complex roots $r = 1 \pm i$. Therefore the complementary solution is

$$y_c(x) = c_1 e^x \sin x + c_2 e^x \cos x$$

Variation of parameters means we look for a solution to the full equation of the form

$$y(x) = u(x)e^x \sin x + v(x)e^x \cos x$$

and solve for the unknown functions $u$ and $v$. We get the following two equations for two unknowns $u'$ and $v'$:

$$\begin{bmatrix}
e^x \cos x & e^x \sin x \\
e^x (\cos x - \sin x) & e^x (\sin x + \cos x)
\end{bmatrix}
\begin{bmatrix}
u' \\
v'
\end{bmatrix} =
\begin{bmatrix}
0 \\
e^x \csc x
\end{bmatrix}$$

We take the determinant (Wronskian $W(x)$) of the matrix on the left, which will help us in finding the solution:

$$W(x) = e^{2x} (\cos x \sin x + \cos^2 x) - e^{2x} (\cos x \sin x - \sin^2 x)$$

$$= e^{2x} (\cos^2 x + \sin^2 x) = e^{2x}$$

Now we use Cramer’s rule to solve for $u'$ and $v'$:

$$u' = \frac{0 - e^x \sin x}{e^{2x} \csc x - e^{2x} (\sin x + \cos x)} = \frac{e^x \sin x}{e^{2x} (\cos x - \sin x) - e^{2x} \csc x}$$

$$v' = \frac{e^x \cos x - 0}{e^{2x} \csc x - e^{2x} (\sin x + \cos x)} = \frac{e^x \cos x}{e^{2x} (\cos x - \sin x) - e^{2x} \csc x}$$

$$u' = -1 \quad u(x) = -x + c_1 \quad v' = \cot x \quad v(x) = \ln |\sin x| + c_2$$

Final solution:

$$y(x) = c_1 e^x \cos x + c_2 e^x \sin x - x e^x \cos x + \ln |\sin x| e^x \sin x$$
7. Find the solution $x(t)$ to the following initial value problem using the Laplace Transform method.

$$x'' + x' - 2x = 6u(t - 2) \quad x(0) = 1 \quad x'(0) = 1$$

**SOLUTION:**

Take the Laplace transform of both sides:

$$\mathcal{L}\{x''\} + \mathcal{L}\{x'\} - 2\mathcal{L}\{x\} = 6\mathcal{L}\{u(t - 2)\}$$

$$[s^2 X - sx(0) - x'(0)] + [sX - x(0)] - 2X = 6\frac{e^{-2s}}{s}$$

$$s^2 X - s - 1 + sX - 1 - 2X = 6\frac{e^{-2s}}{s}$$

$$(s^2 + s - 2)X - (s + 2) = 6\frac{e^{-2s}}{s}$$

$$X = e^{-2s} \left[ \frac{6}{s(s + 2)(s - 1)} \right] + \frac{1}{s - 1}$$

A partial fractions expansion shows that

$$\frac{6}{s(s + 2)(s - 1)} = \frac{-3}{s} + \frac{1}{s + 2} + \frac{2}{s - 1}$$

Taking the inverse transform we get

$$x(t) = u(t - 2) \left[ -3 + e^{-2(t - 2)} + 2e^{t - 2} \right] + e^t$$
8. (a) Find a general solution for the pair of functions $x_1(t)$ and $x_2(t)$ that satisfy the following system of differential equations.

$$ x'_1 = x_2 \quad x'_2 = x_1 $$

**SOLUTION:**

In matrix form we have

$$ x' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x $$

Find eigenvalues by solving

$$ \left| \begin{array}{cc} -\lambda & 1 \\ 1 & -\lambda \end{array} \right| = \lambda^2 - 1 = 0 $$

So $\lambda = \pm 1$. For $\lambda_1 = 1$, we solve

$$ \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} v_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \text{ so } v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} $$

For $\lambda_2 = -1$, we solve

$$ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} v_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \text{ so } v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} $$

So we have

$$ x = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} $$

or

$$ x_1(t) = c_1 e^t + c_2 e^{-t} $$

$$ x_2(t) = c_1 e^t - c_2 e^{-t} $$

(b) Find the eigenvalues and corresponding eigenvectors of the following matrix.

$$ \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} $$

**SOLUTION:**

$$ \left| \begin{array}{cc} 3 - \lambda & -2 \\ 4 & -1 - \lambda \end{array} \right| = (3 - \lambda)(-1 - \lambda) + 8 = \lambda^2 - 2\lambda + 5 = 0 $$
So we have eigenvalues $\lambda = 1 \pm 2i$. For $\lambda_1 = 1 + 2i$, we consider

$$
\begin{bmatrix}
2 - 2i & -2 \\
4 & -2 - 2i
\end{bmatrix}
\begin{bmatrix}
v_1
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0
\end{bmatrix}
$$

which is solved by the eigenvector $v_1 = 
\begin{bmatrix}
1 \\
1 - i
\end{bmatrix}$. The eigenvector corresponding to $\lambda_2 = 1 - 2i$ will be the complex conjugate of $v_1$:

$$
v_2 = 
\begin{bmatrix}
1 \\
1 + i
\end{bmatrix}
$$