

# CUBIC SURFACES WITH SPECIAL PERIODS

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**Abstract.** We show that the vector of period ratios of a cubic surface is rational over  $\mathbb{Q}(\omega)$ , where  $\omega = \exp(2\pi i/3)$  if and only if the associate abelian variety is isogeneous to a product of Fermat elliptic curves. We also show how to construct cubic surfaces from a suitable totally real quintic number field  $K_0$ . The ring of rational endomorphisms of the associated abelian variety is  $K = K_0(\omega)$ .

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## 1. INTRODUCTION

Consider an elliptic curve  $E$  with equation

$$(1) \quad y^2 = 4x^3 - g_2x - g_3.$$

The endomorphism ring of such a curve always contains the integer dilations  $z \mapsto nz$ . An elliptic curve with larger endomorphism ring is said to have *complex multiplication*. The additional endomorphisms are of the form  $z \mapsto \lambda z$ , and it is easy to see that  $\lambda$  lies in a purely imaginary quadratic extension  $K = \mathbb{Q}(\sqrt{-d})$  of the rational numbers. This number field can be identified with  $\text{End}(E) \otimes \mathbb{Q}$  and is called *the CM field* of  $E$ . Conversely, if  $K$  is a purely imaginary extension of the rational numbers, an elliptic curve with CM field  $K$  can be constructed as  $\mathbb{C}/\mathcal{O}_K$ , where  $\mathcal{O}_K$  is the ring of integers in  $K$ .

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Now fix a symplectic homology basis  $\{\gamma_1, \gamma_2\}$ , and let

$$\omega_i = \int_{\gamma_i} \frac{dx}{y}$$

be the *fundamental periods*. If we write (1) as

$$y^2 = 4(x - e_1)(x - e_2)(x - e_3),$$

then fundamental periods are the elliptic integrals

$$\omega_i = \int_{e_i}^{\infty} \frac{dx}{\sqrt{(x - e_1)(x - e_2)(x - e_3)}}.$$

A theorem of Siegel (1932) tells us that if the coefficients  $g_i$  are algebraic numbers, then at least one of the periods is transcendental, and in 1934, Schneider showed that the nonzero periods are always transcendental. For a specific example, consider the Fermat elliptic curve, with affine equation  $y^3 = x^3 - 1$ . A fundamental period is given by

$$\pi_1 = \int_1^{\infty} \frac{dx}{\sqrt{x^3 - 1}} = \frac{1}{3}B(1/6, 1/2) = \frac{\Gamma(1/3)^3}{2^{4/3}\pi}.$$

See [6, equation (10)].

When  $E$  has complex multiplication, the *period ratio*

$$\tau = \omega_2/\omega_1.$$

lies in the CM field. In this case, the transcendence degree of the field the periods,  $\mathbb{Q}(\omega_1, \omega_2)$ , is one. It is conjectured that in the non-CM case, the field generated by the periods is of transcendence degree two.

Consider next a cubic surface  $X$ . Because Hodge structure on  $H^2(X)$  is entirely of type  $(1, 1)$ , there are no periods of interest. However, there is an auxiliary Hodge structure of weight three associated to  $X$  which does have interesting periods and which behaves in many ways like the Hodge structure of an elliptic curve. This is the Hodge structure of a three-sheeted cover  $Y$  of projective 3-space branched along  $X$ . The period domain for such Hodge structure is the unit ball in complex four-space. There is a natural period map

$$(2) \quad \mathcal{M}_s \longrightarrow B^4/\Gamma$$

from the moduli space of stable cubic surfaces (those with at worst nodal singularities) to the indicated quotient of the ball by an arithmetic group. It is an isomorphism [1]. The aim of this paper is to explore the relation between cubic surfaces and their periods. The theory of elliptic curves serves as a model of what might be possible to establish, at least in part. In particular, we will investigate the rationality of period ratios. While the results presented in this direction are modest, they point to several interesting questions. We discuss them at the end of this paper.

We are indebted to Madhav Nori, who explained to us how to use the different to check whether the Abelian variety defined by a number field is principally polarized.

## 2. STATEMENT OF RESULTS

Let  $\sigma$  be one of the two automorphisms of  $Y$  over  $\mathbb{P}^3$ . According to the main result of [1], the pair  $(H^3(Y), \sigma)$ , which we shall call an *Eisenstein Hodge structure*, determines  $X$  up to isomorphism.

Let us elaborate on the previous statement. Since  $H^{3,0}(Y) = 0$ , the Hodge structure has the form  $H^{2,1} \oplus H^{1,2}$ . Therefore the intermediate Jacobian of  $Y$  is an abelian variety endowed with the automorphism  $\sigma$ . The dimension of  $H^{2,1}$  is five. Principally polarized Hodge structures of level one of this “genus” are parametrized by the Siegel upper half space  $\mathcal{H}_5$  modulo the action of the group of ten by ten integer matrices which preserve the standard symplectic form. This space has dimension fifteen.

The subspace of isomorphism classes of Eisenstein Hodge structures has dimension four and is parametrized by a quotient  $B^4/\Gamma$  of the unit ball in complex four-space by an arithmetic group. The group  $\Gamma$  is easy to describe. Let  $h = -|Z_0|^2 + |Z_1|^2 + \cdots + |Z_4|^2$  be the standard hermitian form of signature  $(4, 1)$  on  $\mathbb{C}^5$ . Fix a primitive cube root of unity  $\omega = \exp 2\pi\sqrt{-1}/3$ . Then  $\Gamma$  is the group of  $h$ -unitary matrices with coefficients in the ring of Eisenstein integers,  $\mathbb{Z}[\omega]$ . The space  $B^4/\Gamma$ , which is geodesically imbedded in a quotient of  $\mathcal{H}_5$  is a Shimura variety.

To state the main result of this paper, we recall how the map (2) is determined by a vector in  $\mathbb{C}^5$ , the *period vector* of  $Y$ . To begin, note that the endomorphism  $\sigma$  makes  $H_3(Y, \mathbb{Z})$  into a free rank five module for the ring of Eisenstein integers. This module carries a natural hermitian of form of signature  $(4, 1)$  and determinant one given by

$$2h(x, y) = \langle (\sigma - \sigma^{-1})x, y \rangle - (\omega - \omega^{-1}) \langle x, y \rangle,$$

Let  $\gamma = \{\gamma_0, \dots, \gamma_4\}$  be a basis for  $H_3(Y, \mathbb{Z})$  as a  $\mathbb{Z}[\omega]$ -module for which the hermitian form is diagonal with diagonal entries  $(-1, +1, +1, +1, +1)$ . The symmetry  $\sigma$  is an automorphism of Hodge structures with eigenvalues  $\omega = 2\pi\sqrt{-1}$  and  $\bar{\omega}$ . Let  $H_\omega^3 \oplus H_{\bar{\omega}}^3$  be the eigenspace decomposition. Because it is compatible with the Hodge decomposition, one has the complex Hodge structure

$$H_\omega^{2,1} \oplus H_{\bar{\omega}}^{1,2}.$$

The summands have dimensions one and four, and  $\sigma$  can be chosen so the first summand has dimension one. Let  $\Phi$  be a nonzero element of  $H_\omega^{2,1}$ . Then the *period vector* of  $H^3(Y)$  is

$$(3) \quad v = \left( \int_{\gamma_0} \Phi, \dots, \int_{\gamma_4} \Phi \right).$$

Let  $f(x, y, z) = 0$  be an affine equation for the cubic surface, in one of the standard affine open sets of  $\mathbb{P}^3$ . Then  $w^3 = f(x, y, z)$  is an affine equation for the cubic threefold. As noted in [1], a natural generator for  $H_{\bar{\omega}}^{2,1}(Y)$  is given by the form

$$\Phi = \frac{dx \wedge dy \wedge dz}{w^4}.$$

This is the analogue of the abelian differential  $dx/y$  for an elliptic curve. If  $f$  has coefficients in a field  $K$ , then so does  $\Phi$  — it is a  $K$ -rational differential. Consequently  $\Phi$  is well-defined up to a nonzero element of  $K$ . Therefore questions about the rationality of periods, referred to such a  $K$ -rational differential, make sense for any field  $L$  extending  $K$ .

There is a philosophy that periods of  $K$ -rational differentials are almost always transcendental; this is, however, difficult to prove in any concrete instance. Thus our focus will therefore be on *period ratios* such as  $\omega_2/\omega_1$  for cubic curves or

$$v_i = \frac{\int_{\gamma_i} \Phi}{\int_{\gamma_0} \Phi}$$

for cubic surfaces. In the last ratio, we may rescale  $\Phi$  so that  $v_0 = 1$ . The resulting periods  $v_i$ , which are now relative to a differential with unknown rationality properties, should be thought of as period ratios. This will be our point of view in what follows.

Let us write the period vector as  $v = (a, b_1, b_2, b_3, b_4) = (a, b)$ . When  $a = 1$ , we say that the period vector is *normalized*. In this case  $b$  is a vector in  $\mathbb{C}^4$  of length at most 1. The corresponding Hodge structure of level one and genus five is determined by a  $5 \times 10$  matrix of periods  $P = (A, B)$ . It is normalized in the case that  $A$  is the identity (in which case  $B$  is symmetric with positive-definite imaginary part). In section 3 we show that in the presence of the action by  $\sigma$ , the period vector and the period matrix determine each other, up to natural equivalences.

Finally, recall that an abelian variety over  $\mathbb{C}$  is said to be of CM-type if it is isogenous to a product  $A_1 \times \cdots \times A_r$  of simple abelian varieties and there are fields  $K_i \subset \text{End}(A_i) \otimes \mathbb{Q}$  such that  $[K_i : \mathbb{Q}] \geq 2 \dim A_i$  (in which case  $[K_i : \mathbb{Q}] = 2 \dim A_i$  and  $K_i = \text{End}(A_i) \otimes \mathbb{Q}$ ). If the fields  $K_i$  are equal, we say that  $K = K_i$  is the CM field of the abelian variety. See Mumford [4, p. 347]. We can now state our first result.

**Theorem 1.** *Let  $X$  be a cubic surface and let  $J$  be its abelian variety. The following are equivalent: (a) one (and hence all) normalized period vectors of  $X$  have coefficients in  $\mathbb{Q}(\omega)$ ; (b) one (and hence all) normalized period matrices of  $X$  have coefficients in  $\mathbb{Q}(\omega)$ , (c)  $J$  is isogenous to a product of Fermat elliptic curves.*

It follows from the theorem that for a cubic surface with period vector in  $\mathbb{Q}(\omega)$ ,  $\text{End}(J) \otimes_{\mathbb{Z}} \mathbb{Q}$  is the ring of  $5 \times 5$  matrices with coefficients in  $\mathbb{Q}(\omega)$ ; as

a consequence of the main theorem, we see that cubic surfaces with period vector in  $\mathbb{Q}(\omega)$  are of CM-type with CM field  $\mathbb{Q}(\omega)$ ,

**Corollary 1.** *The Hodge structures of CM type in  $\mathbb{C}H^4$  are dense.*

Because the period map (2) is surjective, there is a dense set of smooth cubic surfaces with periods in  $\mathbb{Q}(\omega)$ . As noted in [1, Theorem 11.6, 11.9], some explicit surfaces with periods of this kind are known. The period vector of the Fermat cubic surface  $x^3 + y^3 + z^3 + w^3 = 0$  is

$$v = (2 - \bar{\omega}, 1, 1, 1, 1).$$

and that of the diagonal cubic surface  $x^3 + y^3 + z^3 + w^3 + u^3 = 0, x + y + x + w + u = 0$  is

$$v = (3, 1, 1, 1, 1)$$

As remarked above, these normalized periods, which are relative to unknown rationality propoerties, should be thought of as period ratios.

It is natural to ask how to characterize cubic surfaces with periods in  $\mathbb{Q}(\omega)$  in purely geometric terms. We do not know the answer, even conjecturally. The last result concerns cubic surfaces whose CM algebra is a field:

**Theorem 2.** *There exist simple, principally polarized abelian varieties  $A$  of dimension five with three-fold symmetry  $\sigma$ . The ring  $\text{End}(A) \otimes \mathbb{Q}$  is isomorphic to a CM-field  $K$  of the form  $K_0(\omega)$ , where  $K_0$  is a totally real field of degree five. The Hodge structure of  $A$  is rational over  $K$ , and it is defined by a period vector rational over the same field.*

### 3. CUBIC SURFACES WITH PERIOD VECTOR RATIONAL OVER $\mathbb{Q}(\omega)$

The key to the proof of the first theorem is an analysis of the periods of the most singular stable cubic surface. We begin with the following:

**Proposition 1.** *Any normalized period vector of the Cayley cubic surface,*

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{w} = 0.$$

*is  $\Gamma$ -equivalent to  $v = (1, 0, 0, 0, 0)$ .*

The Cayley cubic surface is the cubic surface with the maximum number (four) of nodes. Any two such surfaces are isomorphic over the complex numbers. As we show in a moment, the period vector determines the period matrix:

**Corollary 2.** *The normalized period matrix of the Cayley cubic surface is  $\Gamma$ -equivalent to*

$$(4) \quad P = (I, \omega I).$$

where  $I$  is the identity matrix. Thus the intermediate Jacobian of the Cayley cubic surface is the product of five Fermat elliptic curves.

As a general principle, the period vector in our context determines the period matrix and conversely:

**Lemma 1.** *The Eisenstein Hodge structure  $(H, \sigma)$  is determined by the complex Hodge structure on  $H_{\bar{\omega}}$ , and vice versa.*

*Proof.* The orthogonal complement of the period vector  $v$  with respect to the hermitian form of  $H_{\bar{\omega}}^{1,2}$  is  $v^{\perp} = H_{\bar{\omega}}^{1,2}$ . Then

$$H^{2,1} = \mathbb{C}v \oplus \overline{v^{\perp}},$$

where the complex conjugation is defined on  $H_{\mathbb{C}}$ , fixing the points of  $H_{\mathbb{R}}$ . Thus the period vector of the complex Hodge structure  $H_{\omega}$  determines the Eisenstein Hodge structure  $(H, \omega)$ . The converse is clear.  $\square$

Let us now compute the period matrix  $P$  of a general cyclic cubic surface. To this end, recall that  $\gamma = \{\gamma_0, \dots, \gamma_4\}$  is a unitary basis of  $H_3(Y, \mathbb{Z})$  as a hermitian Eisenstein module. Let  $\gamma' = \{\gamma'_0, \dots, \gamma'_4\} = \{\sigma^{-1}\gamma_0, \sigma\gamma_1, \dots, \sigma\gamma_4\}$ . The the homology basis  $\gamma \cup \gamma'$  is a symplectic basis with respect to the standard form

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

where  $I$  is the identity matrix. Choose a basis  $\{\Phi^0, \dots, \Phi^4\}$  for  $H^{2,1}$  where  $\Phi^0 \in H_{\bar{\omega}}^{2,1}$  and  $\Phi^i \in H_{\omega}^{2,1}$  for  $i > 0$ . Then the period matrix takes the form

$$P = (A, B)$$

where

$$A_{ij} = \int_{\gamma_j} \Phi^i \quad B_{ij} = \int_{\gamma'_j} \Phi^i$$

The change of variable formula in the calculus coupled with the fact that the  $\Phi^i$  are eigenvectors of  $\sigma$  imply that

$$(5) \quad \int_{\gamma'_j} \Phi^i = \lambda \int_{\gamma_j} \Phi^i,$$

where  $\lambda \in \{\omega, \bar{\omega}\}$ . We may choose the basis  $\{\Phi^1, \dots, \Phi^4\}$  so that  $A_{ij} = \delta_{ij}$  for  $i, j \in \{1, \dots, 4\}$ . The first Riemann bilinear relation determines the first column of  $A$  in terms of the second, while (5) determines  $B$  in terms of  $A$ .

We conclude that

$$(6) \quad P = \begin{pmatrix} 1 & b_1 & b_2 & b_3 & b_4 & \omega & \bar{\omega}b_1 & \bar{\omega}b_2 & \bar{\omega}b_3 & \bar{\omega}b_4 \\ b_1 & 1 & 0 & 0 & 0 & \bar{\omega}b_1 & \omega & 0 & 0 & 0 \\ b_2 & 0 & 1 & 0 & 0 & \bar{\omega}b_2 & 0 & \omega & 0 & 0 \\ b_3 & 0 & 0 & 1 & 0 & \bar{\omega}b_3 & 0 & 0 & \omega & 0 \\ b_4 & 0 & 0 & 0 & 1 & \bar{\omega}b_4 & 0 & 0 & 0 & \omega \end{pmatrix}$$

In the case of the Cayley cubic surface, the parameter vector  $b$  is zero and the period matrix is

$$P = (I, \omega I),$$

where  $I$  is the identity matrix. It follows that the abelian variety of the Cayley cubic surface is the product of five Fermat elliptic curves.

The map  $\psi(b) = Z$ , where  $Z = BA^{-1}$  is a matrix period quotients, gives an imbedding of the unit ball in the Siegel upper half space of genus five. It can be written more concisely as

$$P(b) = \begin{pmatrix} 1 & b & \omega & \bar{\omega}b \\ \mathfrak{b} & I & \bar{\omega}\mathfrak{b} & \omega I \end{pmatrix}$$

To understand better the location of  $Z(b)$  in the Siegel upper half space, note the quantity  $\delta = \det A = 1 - b \cdot b$  is nonzero since  $|b|^2 < 1$ . Thus, if  $P = (A, B)$  is the period matrix, we can form  $A^{-1}(A, B) = (1, Z)$ , where  $Z$  is the normalized matrix of  $B$ -periods, a symmetric matrix with positive definite imaginary part. The inverse of  $A$  is given by the following quadratic expression:

$$A^{-1} = \delta^{-1} \begin{bmatrix} 1 & -b_1 & -b_2 & -b_3 & -b_4 \\ -b_1 & \delta_1 & b_1b_2 & b_1b_3 & b_1b_4 \\ -b_2 & b_1b_2 & \delta_2 & b_2b_3 & b_2b_4 \\ -b_3 & b_1b_3 & b_2b_3 & \delta_3 & b_3b_4 \\ -b_4 & b_1b_4 & b_2b_4 & b_3b_4 & \delta_4 \end{bmatrix},$$

where  $\delta = 1 - (b_1^2 + b_2^2 + b_3^2 + b_4^2)$  is the determinant of  $A$  and where  $\delta_i = \delta + b_i^2$ . Then

$$(7) \quad Z = A^{-1}B = \delta^{-1} \begin{bmatrix} \delta'_0 & -\theta b_1 & -\theta b_2 & -\theta b_3 & -\theta b_4 \\ -\theta b_1 & \delta'_1 & \theta b_1b_2 & \theta b_1b_3 & \theta b_1b_4 \\ -\theta b_2 & \theta b_1b_2 & \delta'_2 & \theta b_2b_3 & \theta b_2b_4 \\ -\theta b_3 & \theta b_1b_3 & \theta b_2b_3 & \delta'_3 & \theta b_3b_4 \\ -\theta b_4 & \theta b_1b_4 & \theta b_2b_4 & \theta b_3b_4 & \delta'_4 \end{bmatrix},$$

where  $\theta = \omega - \omega^{-1} = \sqrt{-3}$ ,

$$\delta'_0 = \omega - \bar{\omega}(b_1^2 + b_2^2 + b_3^2 + b_4^2),$$

and

$$\delta'_i = \omega\delta - \theta b_i^2.$$

The matrix  $Z$  can be rewritten as

$$(8) \quad Z(b) = \omega I + (\theta/\delta) \begin{pmatrix} b \cdot b & -b \\ -\mathfrak{b} & b \otimes b \end{pmatrix},$$

where  $b$  is viewed as a row vector,  $\mathfrak{b}$  is the corresponding column vector, and  $b \otimes b$  is the matrix whose  $ij$ -th entry is  $b_i b_j$ . Thus the quadratic function

$$\psi(b_1, \dots, b_4) = Z(b)$$

maps points of the unit ball to points of the Siegel upper half space, with the origin of the ball mapped to the normalized period matrix of product of five Fermat elliptic curves.

**Proof of Theorem 1.**

The proof consists of the following three steps. (1) The isogeny class of an abelian variety with lattice  $\Lambda \subset \mathbb{C}^n$  is the isomorphism class of the embedding  $\Lambda \otimes \mathbb{Q} \subset \mathbb{C}^n$  (isomorphism by complex linear maps of  $\mathbb{C}^n$ ). (2) If  $\Lambda_0$  is the lattice of  $E^5$ , where  $E$  is the Fermat elliptic curve, then  $\Lambda_0 \otimes \mathbb{Q} = \mathbb{Q}(\omega)^5 \subset \mathbb{C}^5$ . (3) If the period vector  $a$  is in  $\mathbb{Q}(\omega)$ , then formula (6) for the period matrix shows the columns of the matrix, which give a basis for the lattice  $\Lambda$ , namely, have entries in  $\mathbb{Q}(\omega)$ . Thus  $\Lambda \otimes \mathbb{Q}$  is isomorphic to  $\Lambda_0 \otimes \mathbb{Q}$  as rational subspaces of  $\mathbb{C}^5$ .

#### 4. SMOOTH CUBIC SURFACES AND QUINTIC NUMBER FIELDS

As we have just seen, one can distinguish certain points in the ball quotient, e.g., those whose period vector is rational over  $\mathbb{Q}(\omega)$ . In that case the Abelian variety is isogeneous to a product of Fermat elliptic curves.

We now seek special points in the ball quotient where the corresponding abelian variety is simple and where the rational endomorphism ring is a number field. To this end, consider first a totally real number field  $K_0$  of degree  $n$ , and let  $K = K_0(\mu)$ . The field  $K$  has  $n$  distinct embeddings  $\tau_i$  in the complex numbers. A *CM type* for  $K$  is a choice  $\Phi = (\tau_1, \dots, \tau_n)$ , where  $\tau_i \neq \bar{\tau}_j$  for any  $i, j$ . Let  $\mathcal{O}_K$  be the ring of integers in  $K$ . Then one can form the complex torus  $A(K, \Phi) = \mathbb{C}^n / \Phi(\mathcal{O}_K)$ . Note that the ring  $\mathcal{O}_K$  acts by endomorphisms on  $A(K, \Phi)$ , so that

$$(9) \quad K \subset \text{End}(A(K, \Phi)) \otimes \mathbb{Q}.$$

If  $A(K, \Phi)$  is simple, then the rational endomorphism ring is a division algebra of dimension at most  $2n$  over  $\mathbb{Q}$ . But  $\dim_{\mathbb{Q}} K = 2n$ , so

$$K = \text{End}(A(K, \Phi)) \otimes \mathbb{Q}.$$

Thus the rational endomorphism ring of such a torus is the field  $K$ .

To polarize the torus  $A(K, \Phi)$ , we follow an argument of Mumford, [3, page 212]. There he claims

(\*) the existence of an element  $\alpha$  of  $K$  such that  $\tau_i(\alpha) = \sqrt{-1}\beta_i$ , where the  $\beta_i$  are positive reals.

Given such an element, the expression

$$(10) \quad H(x, y) = 2 \sum_{i=1}^g \beta_i \tau_i(x) \overline{\tau_i(y)}.$$

defines is a positive Hermitian form. Its associated skew form is

$$(11) \quad \Omega(x, y) = \Im H(x, y) = -2\Re \sum_{i=1}^g \tau_i(\alpha) \tau_i(x) \overline{\tau_i(y)} = -\text{Tr}_{K/\mathbb{Q}}(\alpha x \bar{y}).$$

Since  $\alpha$  is an element of the field  $K$ , the trace form takes rational values. To ensure that the values of the form are integers, we choose  $\alpha$  to be in  $\mathcal{O}_K^\vee$ , the lattice dual to  $\mathcal{O}_K$ .

To show the existence of the element  $\alpha$  in (\*), Mumford argues as follows. Let  $\delta$  be an element of  $K_0$  such that  $K = K_0(\sqrt{\delta})$ . Since  $K$  is a totally imaginary extension of the totally real field  $K_0$ , for each  $i$  we can write  $\tau_i(\sqrt{\delta}) = \sqrt{-1}\gamma_i$ , where  $\gamma_i$  is a nonzero real number. One can also find an element  $\eta \in K_0$  such that  $\tau_i(\eta)$  and  $\gamma_i$  have the same sign for all  $i$ . Then  $\alpha = \eta\sqrt{\delta}$  is such that  $\tau_i(\alpha) = \sqrt{-1}\beta_i$ , where  $\beta_i$  is a positive real number for all  $i$ . To summarize, we have the following:

**Proposition 2.** *Given an element  $\alpha \in \mathcal{O}_K^\vee$  satisfying (\*), the expression (10) defines a polarization of  $A(K, \Phi)$ .*

There remains the question of whether this polarization, which is determined by a suitable element  $\alpha \in \mathcal{O}_K$ , can be chosen to be principal. To his end, let  $\mathcal{O}_K^\vee$  be the lattice dual to  $\mathcal{O}_K$  with respect to the trace pairing, and suppose that  $\alpha\mathcal{O}_K \subset \mathcal{O}_K^\vee$ . Then  $\Omega(x, y)$  takes values in  $\mathbb{Z}$ , the ring of rational integers. If in addition

$$(12) \quad \alpha\mathcal{O}_K = \mathcal{O}_K^\vee,$$

then the form  $\Omega(x, y)$  is unimodular. In exactly this case the abelian variety  $A(K, \Phi)$  is principally polarized. The condition  $\alpha\mathcal{O}_K = \mathcal{O}_K^\vee$  holds whenever  $\alpha^{-1}$  generates the *different* of  $K$ , that is, the fractional ideal  $(\mathcal{O}_K^\vee)^{-1}$ . (Thus in the cases we consider, the different is a principal ideal.) To conclude, we have

**Proposition 3.** *The polarization defined by  $\alpha \in \mathcal{O}_K^\vee$  is principal if  $\alpha^{-1}$  generates the different of  $K$ .*

In the case that  $K = K_0(\theta)$ ,  $\theta^2 = -3$ , one can restate the criterion (\*) in terms of the different of  $K_0$ . To this end we introduce the following notion.

**Definition 1.** *Let  $\beta$  be an element of a totally real field  $K_0$  of degree  $n$  over  $\mathbb{Q}$ . Let  $\tau_i$ ,  $i = 1..n$  be the imbeddings of  $K_0$  in the real numbers. Let  $\epsilon$  be an  $n$ -vector with entries  $\pm 1$ . Then  $\beta$  is  $\epsilon$ -positive if*

$$\epsilon_i \tau_i(\beta) > 0 \text{ for all } i.$$

In the case that  $\epsilon_i = +1$  for all  $i$ ,  $\epsilon$ -positivity the same as total positivity.

**Proposition 4.** *Let  $\Phi$  be a CM-type for  $K$  which extends the CM-type for  $K_0$  with  $\tau_i(\theta) = \epsilon_i \theta$ ,  $\epsilon_i = \pm 1$ . Let  $\beta$  be an  $\epsilon$ -positive element of  $K_0$  whose*

inverse generates the different of  $K_0$ . Then  $\alpha = -\beta\theta^{-1}$  defines a principal polarization of  $A(K, \Phi)$ .

*Proof.* The element  $\alpha = -\beta\theta^{-1}$  is the product of inverses of generators of the different for  $K_0$  and  $\mathbb{Q}(\omega)$ . By the multiplicativity of the different, this product is an inverse of a generator of the different of  $K$ , that is, a generator of  $\mathcal{O}_K^\vee$ . See [5, Prop. 2.2, p. 195]. Now

$$\tau_i(\alpha) = -\tau_i(\beta)\tau_i(\theta^{-1}) = \sqrt{-1} \left( \epsilon_i \tau_i(\beta) / \sqrt{3} \right).$$

The quantity in parenthesis be positive if  $\epsilon_i \tau_i(\beta)$  is positive, as asserted.  $\square$

There remains the question of whether the conditions of  $\epsilon$ -positivity for a generator of the different of  $K_0$  can ever be satisfied. The next result sets forth a criterion for its satisfaction which can sometimes be verified by computation. We will carry out such a computation in the next section.

**Proposition 5.** *Let  $\Phi$  and  $\beta$  be as in the preceding proposition. Suppose that there is a unit  $\eta$  of  $\mathcal{O}_{K_0}$  such that  $\tau_i(\eta)$  has the same sign as  $\epsilon_i \tau_i(\beta)$ . Then the element  $\beta\eta$  is  $\epsilon$ -positive.*

*Proof.* If the vectors  $(\tau_1(\eta), \dots, \tau_n(\eta))$  and  $(\epsilon_1 \tau_1(\beta), \dots, \epsilon_n \tau_n(\beta))$ , lie in the same octant then the conditions of the previous proposition are satisfied.  $\square$

Let us return to the case of  $K = K_0(\theta)$ , where  $K_0$  is a totally real quintic field, and consider the problem of whether there are choices so that  $A(K, \Phi, \alpha)$  is a principally polarized abelian variety which has an automorphism of order three, and such that the eigenvalues of that automorphism on the space of abelian differentials are  $\omega$  with multiplicity four and  $\bar{\omega}$  with multiplicity one.

To this end, let  $\tau_1, \dots, \tau_5$  be the imbeddings of  $K_0$  in  $\mathbb{R}$ . Extend them to imbeddings of  $K$  in  $\mathbb{C}$  by requiring

$$(13) \quad \tau_1(\omega) = \bar{\omega} \text{ and } \tau_i(\omega) = \omega \text{ for } i > 1.$$

Then  $(\tau_1, \dots, \tau_5)$  is a CM type for  $K$  with the property that  $\Phi(\omega)$  acts on  $\mathbb{C}^5$  with eigenvalues  $(\bar{\omega}, \omega, \omega, \omega, \omega)$ . It follows that

$$\dim H_{\bar{\omega}}^{1,0}(A(K, \Phi)) = 1.$$

Now let us construct a field with required properties. Let  $\zeta = \exp(2\pi\sqrt{-1}/p)$  be a primitive  $p$ -th root of unity. The totally real subfield of  $\mathbb{Q}(\zeta)$  is  $K_0 = \mathbb{Q}(\rho)$ , where

$$\rho = \zeta + \zeta^{-1} = 2 \cos(2\pi/11) = 1.6825\dots$$

It is the totally really quintic field of smallest discriminant, namely  $11^4 = 14641$ . It is not hard to see that  $\mathbb{Q}(\zeta)$  is a CM field and that one can construct a simple Abelian variety from it. See [2, p. 24]). What we need,

however, is a simple abelian variety with a suitable action of  $\omega$ . To this end, we establish the following:

**Proposition 6.** *Let  $K = K_0(\omega)$ . Let  $\Phi$  be the CM type extended from a CM type  $(\tau_1, \dots, \tau_5)$  of  $K_0$  as in equation (13). Then  $A(K, \Phi)$  is a simple, principally polarized abelian variety. It therefore corresponds to a smooth cubic surface.*

*Proof.* The class number of  $K_0$  is one, so that all ideals in  $\mathcal{O}_{K_0}$  are principal. The different of  $\mathcal{O}_{K_0}$  is the ideal generated by

$$\delta_0 = -4r^4 + r^3 + 14r^2 + 4r - 9,$$

where  $r = -\rho$  generates  $K_0$ . This element is not totally positive. The group of units of  $\mathcal{O}_{K_0}$  is isomorphic to  $C_2 \times \mathbb{Z}^4$ . Generators for the free abelian part of this group are

$$u_1, u_2, u_3, u_4 = r^4 - 3r^2 + 1, r^2 - r - 1, r - 1, r$$

One finds that element

$$\delta = r\delta_0,$$

which also generates the different ideal, is  $\epsilon$ -positive for  $\epsilon = (+, +, -, +, +)$ . The element  $\beta = \delta^{-1}$  is also  $\epsilon$ -positive, and it is the element we use to define the principal polarization. The embeddings of  $\delta$  in  $\mathbb{R}$  are

21.7307463515808  
 4.26952134163076  
 -1.91569396353523  
 14.0542888631537  
 5.86113740717001

The corresponding embeddings of the generator  $r$  of  $K$  over  $\mathbb{Q}$  are

-1.68250706566236  
 -0.830830026003773  
 0.284629676546570  
 1.30972146789057  
 1.91898594722899

The criterion for  $\epsilon$ -positivity is satisfied, and the abelian variety  $A(K, \Phi, \alpha)$  is principally polarized.

To conclude that  $A(K, \Phi, \alpha)$  corresponds to a smooth cubic surface, we must show that it is irreducible. It will be enough to show that it is simple. To that end, consider the Galois group  $G$  of  $K/\mathbb{Q}$ . It is a group of order ten. The subgroup  $Gal(K_0/\mathbb{Q})$  is normal. Since it is of order five, it is cyclic. The subgroup  $Gal(\mathbb{Q}(\omega)/\mathbb{Q})$  is also normal, and since it is of order two, it is cyclic. A group of order ten with these structural features must be cyclic.

The subfields of  $K$  are the fixed sets of subgroups of the Galois group. Since a cyclic group of order ten has only two non-trivial subgroups, the field  $K$  has only two nontrivial subfields, namely,  $K_0$  and  $\mathbb{Q}(\omega)$ . Once we know the subfields of  $K$ , Mumford's criterion for simplicity of  $A(K, \Phi, \alpha)$  [3, p. 213-14] is easy to apply. It suffices to show that there are  $\tau_i$  and  $\tau_j$  such that  $\tau_i|_{\mathbb{Q}(\omega)} \neq \tau_j|_{\mathbb{Q}(\omega)}$ . The condition on the  $\tau$ 's holds by construction. See equation (13).  $\square$

We have a construction

$$\{\text{Certain totally real quintic number fields}\} \longrightarrow \{B^4/\Gamma\}$$

The period matrices that arise in this way are rational over  $K$ . Indeed, consider the formula (7) for the normalized period matrix  $Z = (Z_{ij})$ , where  $i$  and  $j$  run from 0 to 4. Then the period vector is

$$b = (1/\theta)(Z_{01}, Z_{02}, Z_{03}, Z_{04})$$

If  $Z_{ij}$  is rational over  $K$  then so is  $b$ . The converse also comes from formula (7). The period vector could be rational over  $K_0$  and still correspond to a period matrix rational over  $K$ ; it cannot, however, be rational over  $\mathbb{Q}$  or  $\mathbb{Q}(\omega)$ .

## 5. COMPUTATIONS AND EXPERIMENTS

Above we described a method to show that the totally real quintic field of discriminant 14641 defines a principally polarized abelian variety with three-fold symmetry of the correct type. The same method can be used to produce lists of quintic fields with this property. Using the Sage code below, for example, we show that (1) there are 414 totally real quintic fields of discriminant less than  $10^6$ ; (2) none of these have class number bigger than 1; (3) there are at least 370 fields which satisfy the hypotheses of Proposition 4 (4) that is, the density of such fields in the indicated range is about 0.89. Below is the data for the fields of discriminant  $< 10^5$ . The second column is the discriminant of the field. The number in the last column is 1 if the field passes the test, 0 if it fails.

	discr	result
1	14641	1
2	24217	1
3	36497	1
4	38569	0
5	65657	1
6	70601	1
7	81509	1
8	81589	1
9	89417	1

**Sage code.** We first define a function to return the class number of the field  $K_0 = \mathbb{Q}[X]/(F[X])$ :

```
def classNumber(F):
    R.<x> = PolynomialRing(QQ)
    f = R(F)
    K.<a> = NumberField(f)
    return K.class_number()
```

Next, we define the function `test(F, MAX)`. If it returns 1, then  $K_0$  passes the test of having  $\epsilon$ -positive generator of the different for some  $\epsilon$  with a single  $-1$ . In paragraph one of the code, a generator `d0` of the different and generators `u[0]`,  $\dots$ , `u[3]` for the free part of the group of units are found for  $K_0$ .

In the second paragraph, a search is conducted over a small box in the positive octant of  $i, j, k, \ell$  space. At each lattice point we ask whether  $d_0 u_0^i u_1^j u_2^k u_3^\ell$  is  $\epsilon$ -positive. If a lattice point passes the test, the function `test` returns 1. If no lattice point passes the test, the function returns 0.

```
def test(F, MAX):
    R.<x> = PolynomialRing(QQ);
    f = R(F)
    K0.<a> = NumberField(f);
    D0 = K0.different(); d0 = D0.gens_reduced()[0]
    u = K0.units()

    i, j, k, l = 0, 0, 0, 0
    for i in range(0,MAX):
        for j in range(0,MAX):
            for k in range(0,MAX):
                for l in range(0,MAX):
                    dd = d0*u[0]^i*u[1]^j*u[2]^k*u[3]^l
                    if epsilon_positive(dd) == True:
                        return 1
    return 0
```

The test for  $\epsilon$ -positivity is carried out by the function below:

```
def epsilon_positive(x):
    ee = x.complex_embeddings()
    delta = 0.0000001
    neg = 0; pos = 0
    for e in ee:
        if e < -delta:
            neg = neg + 1
        if e > delta:
```

```

    pos = pos + 1
    if neg == 1 and pos == 4:
        return True
    else:
        return False

```

Finally, we enumerate the totally real quintic fields of discriminant less than  $N$ , applying the above test to each, and collecting various statistics.

```

def testFields(N, BOX):
    TRF = enumerate_totallyreal_fields_all(5, N)
    n = 1; nNonUFD = 0; nPass = 0

    for field in TRF:
        discr, G = field
        cn = classNumber(G)
        if (cn == 1):
            result = test(G, BOX)
            if (result == 1):
                nPass = nPass + 1
            else:
                result = -1
                nNonUFD = nNonUFD + 1
        print "%5d %4d %2d" % (n, discr, result)
        n = n + 1

    print "Summary:"
    print "  Number of fields:", len(TRF)
    print "  Number of fields of class number > 1:", nNonUFD
    print "  Number of fields which satisfy the criterion:", nPass

```

To run the test on fields of discriminant  $< 10^6$  with a box of lattice points that measures five units on a side, run the command `testFields(106, 5)`.

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