Introduction to Algebraic and Geometric Topology Week 7

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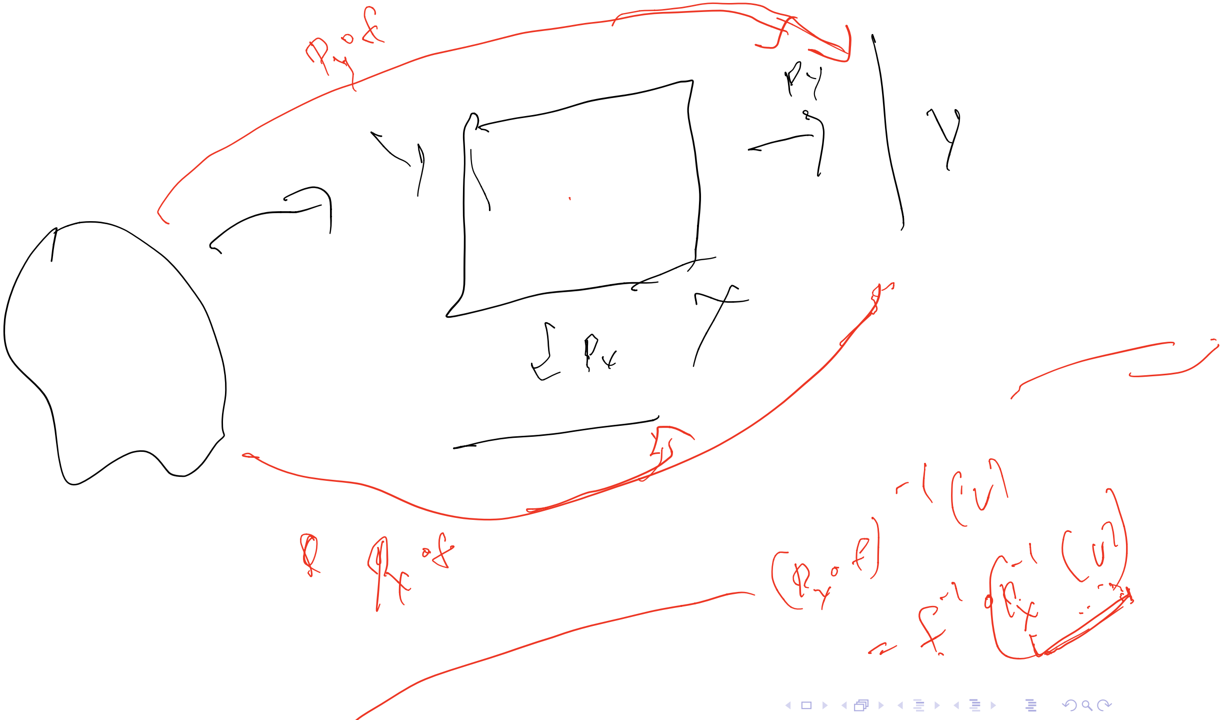
Characterization of Product Topology

- $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ topological spaces.
- $(X \times Y, \mathcal{T}_{X\times Y})$ product topology
- Recall projections

$$p_X : X \times Y \to X \quad \text{and} \quad p_Y : X \times Y \to Y$$

- $(Z, \mathcal{T}_Z)$ a topological space.

- **Theorem**
  A map $f : Z \to X \times Y$ is continuous if and only if both compositions $p_X \circ f$ and $p_Y \circ f$ are continuous.


φ_Y ∘ f
φ_Y
Y
φ_X
X
φ_X ∘ f
(φ_Y ∘ f)^{-1}(V)
= f^{-1} ∘ φ_X^{-1}(V)

- Proof:
- $\Longrightarrow$ clear: composition of continuous maps.
- $\Longleftarrow$: If $p_X \circ f$ is continuous, then for all open $U \subset X$, $V \subset Y$,

$$(p_X \circ f)^{-1}(U) = f^{-1}(U \times Y) \text{ and } (p_Y \circ f)^{-1}(V) = f^{-1}(X \times V)$$

  are open in Z.
- Therefore

$$f^{-1}(U \times V) = f^{-1}((U \times Y) \cap (X \times V))$$

  is open in Z
- Thus f is continuous

$U \subset X$

$V \subset Y$

$(U \times Y) \cap (X \times V)$

Sub-basis for a topology

$= U \times V.$

- The critical property we used of the collection of open sets

$$\{p_X^{-1}(V) \mid V \in \mathcal{T}_Y\} \cup \{p_Y^{-1}(U) \mid U \in \mathcal{T}_X\}$$

  is that it is a *sub-basis* for the topology of $X \times Y$.

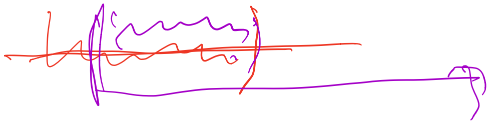
- Definition

  Let $(Z, \mathcal{T}_Z)$ be a topological space. A subset $\mathcal{S} \subset \mathcal{T}_Z$ is called a *sub-basis* for $\mathcal{T}_Z \iff$ every $U \in \mathcal{T}_Z$ is a union of finite intersections of elements of $\mathcal{S}$.

- In other words, the collection of finite intersections of elements of $\mathcal{S}$ forms a basis for $\mathcal{T}_Z$.
- To check that a map $f : (W, \mathcal{T}_W) \to (Z, \mathcal{T}_Z)$ is continuous, enough to check

$$f^{-1}(U) \in \mathcal{T}_W \text{ for all } U \in \mathcal{S}.$$

Examples



- Sub-basis for $X \times Y$
- $\{(a, \infty) \mid \ \in \mathbb{R}\} \cup \{(-\infty, b) \mid b \in \mathbb{R}\}$ is a sub-basis for the topology of $\mathbb{R}$

all open intervals

$\implies$ all open sets

- Similar sub-basis for $\mathbb{R}^2$?

Infinite Products: Review and Correction

- A an index set. → $\mathbb{N}, \mathbb{Z}, \mathbb{R}$

$\alpha \in A$

X_α

- $\{X_\alpha\}_{\alpha \in A}$ a collection of sets indexed by A.

  with $X_\alpha \neq \emptyset$ for all α.
- $\bigcup_{\alpha \in A} X_\alpha$ their union.

  (Rather than the disjoint union $\coprod_\alpha X_\alpha$ as last time).
- The *product* of the X_α is defined as

$$\prod_{\alpha \in A} X_\alpha = \{f : A \to \bigcup X_\alpha \mid \forall \alpha \in A, f(\alpha) \in X_\alpha\}$$

Same α.

Examples

- $A = \{1,2\}$ then

$$\prod_{\alpha \in \{1,2\}} X_\alpha = \{f : \{1,2\} \to X_1 \bigcup X_2 \mid f(1) \in X_1, f(2) \in X_2\}$$

Letting $x_1 = f(1)$ and $x_2 = f(2)$, this is the same as

$$\{(x_1, x_2) \mid x_1 \in X_1, x_2 \in X_2\}$$

$1 \to X_1$

$2 \to X_2$

which is the usual definition of $X_1 \times X_2$.

- Similarly, if $A = \{1, 2, \ldots, n\}$, a finite set, then $\prod_{\alpha \in A} X_\alpha$ is equivalent to the usual definition:
- A function

$$f : \{1, 2, \ldots, n\} \to X_1 \cup X_2 \cup \cdots \cup X_n$$

with

$$f(i) \in X_i \text{ for } i = 1, \ldots, n$$

is equivalent to the n-tuple

$$x_1 = f(1), x_2 = f(2), \ldots x_n = f(n)$$

which is the usual definition

$$X_1 \times X_2 \times \cdots \times X_n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in X_i\}$$

- If A is an arbitrary set, need the *Axiom of choice*.
- One formulation of the axiom:

  $A \neq \emptyset$ and $\forall \alpha \in A, X_\alpha \neq \emptyset$

  $\Longrightarrow$

  $\prod_{\alpha \in A} X_\alpha \neq \emptyset$.

- (The functions $f : A \to \bigcup X_\alpha$ "choose " an element from each X_α)

Main Example (for us)

:

- If $A = \mathbb{N}$, the natural numbers, and $X_\alpha = X$ for all α,

$$\prod_{i \in \mathbb{N}} X = X^{\mathbb{N}} = \{ \text{ Sequences } \{x_i\}_{i \in \mathbb{N}} \}$$

in other words, the collection of all sequences in X.

$\mathbb{N} \to X$

$n \to x_n \in X$

$\{x_i\}$

$X^{\mathbb{N}}$

Topology in Product Space

- Suppose A arbitrary and each X_α has a topology $\mathcal{T}_\alpha$.
- Let $\mathcal{B}_{\prod X_\alpha}$ be defined as follows:
  - For each finite subset $F = \{\alpha_1, \ldots, \alpha_n\} \subset A$ *choose*, for each $\alpha_i \in F$, an open set $U_{\alpha_i} \subset X_{\alpha_i}$.
  - Call this finite collection

$$\mathcal{U}_F = \{U_{\alpha_1}, \ldots, U_{\alpha_n}\}.$$

  - This notation makes the finite set F clear, but leaves the choices of open sets U_{α_i} understood.

$\alpha \in A$

F

$\{\alpha_1, \ldots, \alpha_n\}$

Fix

① $F = \{\alpha_1, \ldots, \alpha_n\} \subset A$

② then $U_\alpha \subset X_\alpha$ $(U_\alpha \in \mathcal{T}_\alpha)$

- To make the choices clear, we should also give a function ϕ that chooses $\phi(\alpha_i) = U_{\alpha_i} \in \mathcal{T}_{X_{\alpha_i}}$, and call the collection $U_{F,\phi}$.
- Will simplify notation to $\mathcal{U}_F$ and leave ϕ understood.

$\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$

$\{\alpha_1, \ldots, \alpha_n\} = F$

- For each $\mathcal{U}_F$, define the set

$$B(\mathcal{U}_F) = \{f \in \prod X_\alpha \mid f(\alpha) \in U_\alpha \text{ for all } \alpha \in F\}$$

$\alpha_1, \ldots, \alpha_n$

- Equivalently,

$$B(\mathcal{U}_F) = \prod_{\alpha \in F} U_\alpha \times \prod_{\alpha \notin F} X_\alpha$$

- Define $\mathcal{B}_{\prod X_\alpha}$ to be the collection of all $B(F, \mathcal{U}_F)$.

Check

- **Theorem**

  $\mathcal{B}_{\prod X_\alpha}$ is a basis for a topology.

- Definition
  The topology $\mathcal{T}_{\prod X_\alpha}$ generated by the basis $\mathcal{B}_{\prod X_\alpha}$ is called the *product topology*

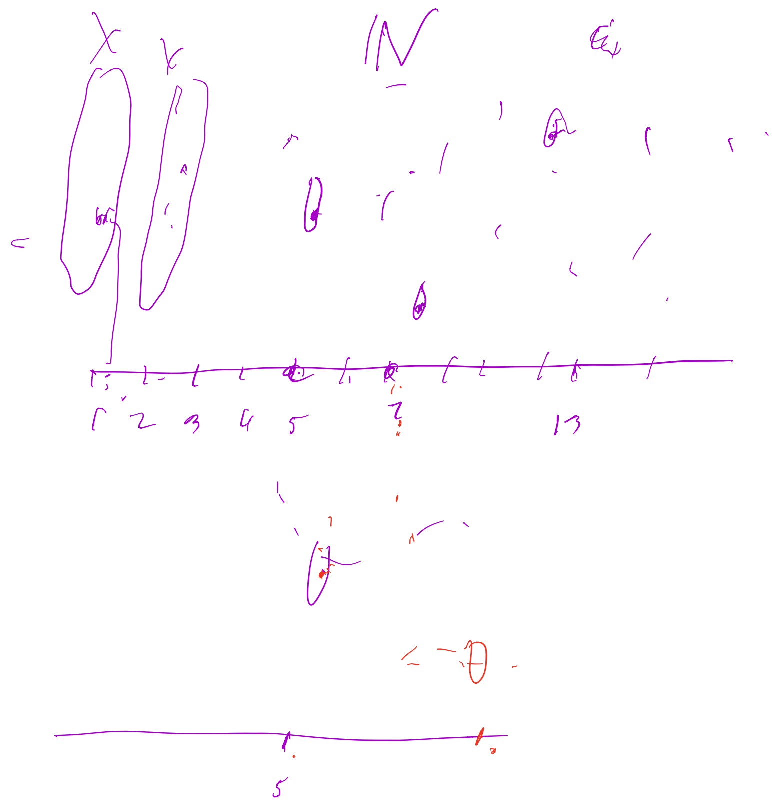
- Essential point: Each basic open set $B(\mathcal{U}_F)$ restricts only finitely many coordinates.

- For A finite get same basis as before.

$X^{\mathbb{N}}$

$F = \{\alpha_1, \dots, \alpha_m\}$

$U_{\alpha_1}, \dots, U_{\alpha_m}$



$\# F = 1$

sub-basis

length.

- For each $\alpha_0 \in A$ can define projection

$$p_{\alpha_0} : \prod_{\alpha \in A} X_\alpha \to X_{\alpha_0}$$

by

$$p_{\alpha_0}(f) = f(\alpha_0)$$

- Recall definition

$$\prod_{\alpha \in A} X_\alpha = \{f : A \to \bigcup X_\alpha \mid \forall \alpha \in A, f(\alpha) \in X_\alpha\}$$

so $p_{\alpha_0}(f)$ is the value of the function f at the element $\alpha_0 \in A$

- As before, the sets $\mathcal{U}_F$ for F a one element set $\{\alpha_1\} \subset A$

$$\mathcal{U}_{\{\alpha_1\}} = \{f : A \to \cup_\alpha X_\alpha \mid f(\alpha_1) \in U_{\alpha_1}\}$$

or, equivalently

$$\mathcal{U}_{\{\alpha_1\}} = U_{\alpha_1} \times \prod_{\alpha \neq \alpha_1} X_\alpha$$

is a *sub-basis* for $\mathcal{T}_{\prod_\alpha X_\alpha}$

- As before we get

- Theorem

  If Z is any topological space,

  a map $f : Z \to \prod_\alpha X_\alpha$ is continuous

  $\iff$

  all compositions $p_\alpha \circ f$ are continuous.

- **Proof.**
  As before: $\Longrightarrow$ clear.
- For $\Longleftarrow$: since the sets $p_\alpha^{-1}(U_\alpha)$ form a sub-basis for the topology of $\prod_\alpha X_\alpha$, we get that f^{-1} of all elements of a sub-basis are open, hence f is continuous.

$\square$

c_0'

- Suppose $A = \mathbb{N}$ and $X_i = X$ for all $i \in \mathbb{N}$..
- Then, given a finite set $F \subset \mathbb{N}$ and for each $i \in F$ a choice of open set $U_i \subset X$, $B(\mathcal{U}_F)$ is the set of all sequences $\{x_i\}_{i=1}^{\infty}$ such that $x_i \in U_i$ for all $i \in F$.

The Cantor Set

- Recall the construction of the Cantor Set $C \subset [0, 1]$:
- Start with the unit interval $[0, 1]$, divide it into three equal intervals, and remove the open middle interval $(\frac{1}{3}, \frac{2}{3})$.
- The space C_1 that remains is the union of two closed intervals;

$$C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$$

- Iterate by applying the same construction to each sub-interval:

- Next step:

$$C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$$

- Continue. At the nth stage we get C_n a union of 2^n intervals.
- Moreover the C_n are nested: $C_1 \supset C_2 \supset \ldots$.
- Consequently

$$\boxed{C = \bigcap_{n=1}^{\infty} C_n \neq \emptyset}$$

- See Figure.

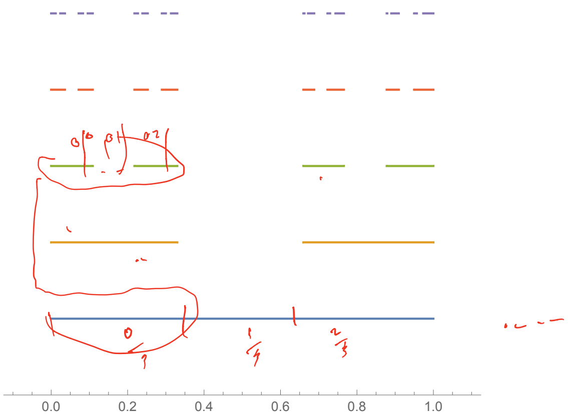


Figure: Constructing the Cantor Set

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Another Descripton of the Cantor Set

- Observe that, by definition of the ternary expansion of a real number

0, 1, 2

$$C = \{ \sum_{i=1}^{\infty} \frac{a_i}{3^i} \mid a_i = 0 \text{ or } 2 \}$$

- Since $a_i = 1$ is not allowed, we get that the map

$$t : \{0, 2\}^{\mathbb{N}} \to C$$

defined by

$$\boxed{t(\{a_i\}) = \sum_{i=1}^{\infty} \frac{a_i}{3^i}}$$

is *bijective*.

$\{0,2\}^{\mathbb{N}} \longrightarrow C[0,1]$

$t : \{0,2\}^{\mathbb{N}} \to [0,1]$

- Give $\{0, 2\}^{\mathbb{N}}$ the product topology.
- Give C the topology as a subspace of $[0, 1]$.

  (the metric topology)

- *Theorem*: t is a homeomorphism.

$\sum^{n} \Big/ \sum$

$< \varepsilon$

- Proof that t is continuous.

$\{0, 2\}$

- Let $a^0 = \{a_i^0\} \in \{0, 2\}^{\mathbb{N}}$ and let $\epsilon > 0$.
- Choose i_0 so that $\frac{1}{3^{i_0}} < \epsilon$

  If $\{a_i\} \in \{0, 2\}^{\mathbb{N}}$ and $a_i = a_i^0$ for $i \leq i_0$, then

$$|t(\{a_i\}) - t(\{a_i^0\})| \leq \sum_{i_0+1}^{\infty} \frac{|a_i - a_i^0|}{3^i} \leq \sum_{i_0+1}^{\infty} \frac{2}{3^i} = \frac{1}{3^{i_0}} < \epsilon$$

- $$U = \{\{a_i\} \in \{0, 2\}^{\mathbb{N}} \mid a_i = a_i^0 \text{ for } i \leq i_0\}$$

  is an open set with $a^0 \in U \subset t^{-1}(B(t(a^0), \epsilon))$.
- Since a^0 and ϵ are arbitrary, t is continuous.

$$\{0,2\}^{\mathbb{N}} \xrightarrow{t} \sum_{i=1}^{\infty} \frac{a_i}{3^i}$$

$$\{a_1, a_2, \ldots\}$$

t homeo $\{0,2\}^{\mathbb{N}} \xrightarrow{\sim} C \subset [0,1]$

Cont at 0.

$t\{\ldots 0\} = 0$

$\forall \varepsilon > 0 \ \exists$ nbd U of $\vec{0}$ in $\prod$ s.t. $|t(x)| < \varepsilon$ $\forall x \in U$

- ~~Proof of continuity of t^{-1}~~: later.

$$|t((a_i))| < \varepsilon \; ; \quad \left| \sum_{i=1}^{\infty} \frac{a_i}{3^i} \right| < \varepsilon$$

$$a_i \le 2$$

$$a_1, \dots, a_n, \; a_{n+1} \dots$$

$$\sum_{n+1}^{\infty} \frac{a_i}{3^i}$$

$$\le \sum_{n+1}^{\infty} \frac{2}{3^i} = \frac{2}{3^{n+1}} \left(\sum_{0}^{\infty} \frac{1}{3^i} \right) = \frac{2}{3^{n+1}} \frac{1}{1 - 1/3}$$

Given $\varepsilon > 0$, Choose n

$1/3^n < \varepsilon$,

$|t\{a_i\}| < \varepsilon$

forget $\{a_1, \dots, a_n\}$ ...

$0 \cdots 0$

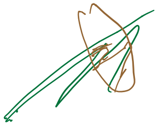
$t\{0, \dots 0, a_{n+1}, a_{n+2} \dots \}| < \varepsilon$

$\frac{1}{3^n}$

Subspace Topology

- $(X, \mathcal{T})$ topological space.
- $A \subset X$
- *Subspace Topology* $\mathcal{T}_A$:

$$\mathcal{T}_A = \{U \cap A \,|\, U \in T_X\}$$



$$= \mathcal{T}_A = \{ i^{-1}(U) : U \in \mathcal{T}_X \}$$

- $\mathcal{T}_A$ is the smallest topology that makes $\iota : A \to X$ continuous.

$\iota : A \to X$ has top $\mathcal{T}_X$

ι cont

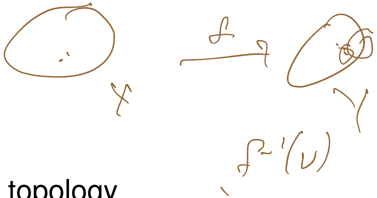
$\Longleftrightarrow \iota^{-1}(U)$ open in A $\forall U$ open in X

$A \to X \supset U$

$\iota^{-1}(U) = \{x \in A : \iota(x) \in U\}$

$= x$

- General Principle
- Let X and Y be sets and let $f : X \to Y$.



1. Given a topology $\mathcal{T}_Y$ on Y there is a smallest topology $\mathcal{T}_X$ on X that makes f continuous,

   namely $\mathcal{T}_X = \{f^{-1}(U) : U \in \mathcal{T}_Y\}$.

2. Given a topology $\mathcal{T}_X$ on X there is a largest topology $\mathcal{T}_Y$ that makes f continuous.,

   namely $\mathcal{T}_Y = \{U \subset Y : f^{-1}(U) \in \mathcal{T}_X\}$.

   (Main case: f is surjective.)

Quotient topology f^{-1}

Examples subspace

- Metric spaces : top of subspace metric
- Open subsets

  $A \subset X$

  U open $A \cap U$ open

  open in X
- Closed subsets

Compact Spaces

- $(X, \mathcal{T})$ called compact $\iff$ every open cover has a finite subcover.

- Open cover of X:

  Collection $\{U_\alpha\}_{\alpha \in A}$ of open sets such that

  $$X = \bigcup_\alpha U_\alpha$$

  cover

- Finite subcover:

  Finite sub-collection $\{U_{\alpha_1}, \dots, U_{\alpha_n}\}$ of $\{U_\alpha\}$ such that

  $$X = U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$$

Examples

Finite

$[0,1] \subset \mathbb{R}$

$X \subset \mathbb{R}^n$ compact $\Longleftrightarrow$ closed & bdd

Compact Subsets

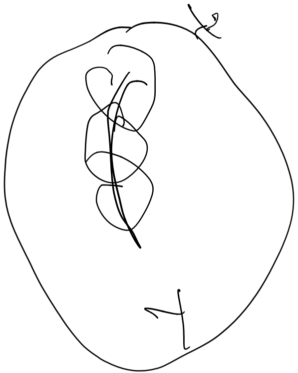
- $Y \subset X$ compact means compact in subspace topology.
- Equivalent formulation in terms of open sets in X:
  - Every open cover of Y, meaning

    Collection $\{U_\alpha\}_{\alpha \in A}$ of open sets in X such that

    $$Y \subset \bigcup_\alpha U_\alpha$$

  - Has a finite subcover, meaning:
    Finite sub-collection $\{U_{\alpha_1}, \dots, U_{\alpha_n}\}$ of $\{U_\alpha\}$ such that

    $$Y \subset U_{\alpha_1} \cup \dots \cup U_{\alpha_n}$$



- Equivalence of two statements:
- By definition of subspace topology:

  Collection $\{V_\alpha\}_{\alpha \in A}$ of open sets in Y
  $\iff$
  Collection $\{V_\alpha\}_{\alpha \in A}$ of open sets in X with $V_\alpha = Y \cap U_\alpha$
- By definition of cover

$$Y = \cup_\alpha V_\alpha = \cup_\alpha (Y \cap U_\alpha)$$

- Which is equivalent to

$$Y \subset \cup_\alpha U_\alpha$$

Properties and consequences of compactness

- A compact metric space is bounded.

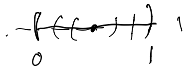
pick $x_0 \in X$, $U_n = B(x_0, n)$

$$\bigcup_{n=1}^{\infty} B(x_0, n) = X$$

$\exists\, x_0 \in X,\ R > 0$ s.t. $X \subset B(x_0, R)$

X cpt $\Rightarrow \exists\, n_1, \dots, n_k$ s.t. $\underbrace{\bigcup B(x_0, n_i)}_{B(x_0, m)} = X$ $\quad m = \max n_1, \dots, n_k$

$\Rightarrow X$ bdd



$\mathbb{R}$

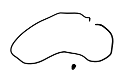
not compact:

$U_n = (1/n,\ 1-1/n)$

$\bigcup U_n = (0,1)$

no finite subcollection ...

C cpct $\Rightarrow C$ closed



$\Leftrightarrow X - C$ is open

$X - \{x\}$

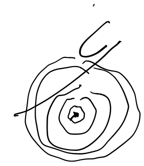
$= \cancel{\{y \in X\}}$

$x \in X - C$

find $r > 0$ s.t. $B(x,r) \cap C = \phi$

$A_n = \{ y \in X : d(x,y) > 1/n \}$

A_n open



$A_n = X - \bar{B}(x, 1/n)$

$\bigcup_{n=1}^{\infty} A_n = \cancel{X - B}$

$= X - \{x\}$

$y \neq x \Rightarrow d(x,y) > 0$

$\Rightarrow \exists\, n \in \mathbb{N}$ s.t. $1/n < d(x,y)$

- A compact subspace of a metric space is closed.

$y \in A_n$

$\{A_n\}$ open cover of $X - \{x\}$

$\supseteq C$

$\Rightarrow$ open cover of C

$\exists$ finite subcover $(*)$ $\Rightarrow n_0 \in \mathbb{N}$

s.t. $B(x, 1/n_0) \cap C = \emptyset$

$\Rightarrow x \in (X - C)^{\circ} \Rightarrow X - C$ open

$\Rightarrow C$ closed.

- A subspace of $\mathbb{R}^n$ is compact if and only if it is closed and bounded (Heine-Borel theorem)

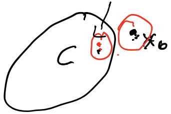
$U \cap V = \phi$

$\forall x, y \in X, \quad x \neq y$

$\exists$ nbhds U of x, V of y,

- A compact subspace of a Hausdorff space is closed.

$x_0 \in X - C$



$\forall x \in C, \exists$

$x \in U_x$ = nbd of x

$x_0 \in V_x$ = nbd of x_0

s.t. $U_x \cap V_x = \phi$

$\{U_x\}_{x \in C}$ open cover of C.

$\Rightarrow$ has finite subcover $U_{x_1}, \dots, U_{x_n}$

$U_{x_i} \cap V_{x_i} = \phi$ $\quad \left(\cap V_{x_i}\right) \cap \left(\cup U_{x_i}\right) = \phi$

nbd of x_0 $\subset$ C $\cup$ $\subset$

- Hausdorff is needed.

$\Rightarrow X - C$ open

C is closed.

$\mathbb{R}, \mathcal{T}_{CF}$ open: ϕ, $\mathbb{R} - F$ F finite.

$x \in U$, $y \in V$ $U \cap V \neq \phi$

$U = (\mathbb{R} - F_1) \cap (\mathbb{R} - F_2) = \mathbb{R} - (F_1 \cup F_2) \neq \phi$

If $A \subset \mathbb{R}, \mathcal{T}_{CF}$ is any subset, A is compact. $\{U_\alpha\}$ open cover.

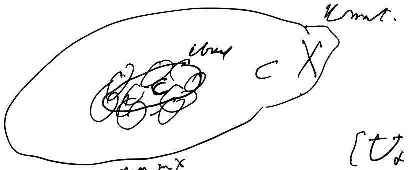
$U_\alpha = \mathbb{R} - F_\alpha$

$A / U_{\alpha_1} \in F_{\alpha_1}$ = finite

finite $U_1, \ldots, U_n$ such

Take $U_2 \ldots U_n$ covering F.

- A closed subspace of a compact topological space is compact.



$\{U_\alpha\}$ open in X, open cover of C.

C closed $\Leftrightarrow$ $X - C$ open.

$\{U_\alpha\}$

$\bigcup U_\alpha \supset C$

$X - C$

$\underbrace{\{U_\alpha\} \cup \{X - C\}}$ covers X

$\Rightarrow$ finite subcover

$U_{\alpha_1}, \ldots, U_{\alpha_n}, (X - C)$ cover X

$\Rightarrow$ $\underbrace{U_{\alpha_1}, \ldots, U_{\alpha_n}}$ cover C

- A continuous image of a compact space is compact.

$f : X \to Y$ cont

$\cup$

C compact

$\Rightarrow f(C)$ compact

$\{U_\alpha\}$ open cover of $f(C)$

$f^{-1}(U_\alpha)$

open cover of C

$\exists\, \alpha_1, \ldots, \alpha_n$

s.t. $f^{-1}(U_{\alpha_1}), \ldots, f^{-1}(U_{\alpha_n})$

cover C.

$\Rightarrow U_{\alpha_1}, \ldots, U_{\alpha_n}$. $f(C)$.

- Another formulation:
- $f : X \to Y$ continuous, $C \subset X$ compact $\Longrightarrow f(C)$ compact.