Introduction to Algebraic and Geometric Topology
Week 4

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Fall 2017

M		W		F
11		13 HW due		A
HW		20		*
		27 midterm		

$E_d(x, y)$

$$E_{d'}(f(x), f(y)) = f(E_d(x,y))$$

if

$$\{z' : d'(f(x), z') + d'(z', f(y)) = d'(f(x), f(y))\}$$

$\exists z$ s.t. $f(z) = z'$

$$\{f(z) : \underbrace{\{d(x,z) + d(z,y) = d(x,y)\}}\}$$

$$\Downarrow$$

$$d'(f(x), f(z)) + d'(f(z), f(y)) \ldots$$

$$f(E_d(x,y)) \subset E_{d'}(fx, fy)$$

z'

Recall:

- (X, d) metric space.
- $\mathcal{I}(X)$ (Or $\mathcal{I}(X, d)$) $=$ the set of all surjective isometries $f : (X, d) \to (X, d)$ forms a *group*.
- Compute $\boxed{\mathcal{I}(\mathbb{R}^2, d) \text{ for } d = d_{(1)}, d_{(2)}, d_{(\infty)}}$.
- Similar answers for $\mathbb{R}^n$, $n > 2$.

Subgroups of $\mathcal{I}(\mathbb{R}^n)$

- Fix a norm on $\mathbb{R}^n$: $|x|_{(1)}$ or $|x|_{(2)}$ or $|x|_{(\infty)}$.
- Let $\mathcal{I} = \mathcal{I}(\mathbb{R}^n, d)$ for $d =$ corresponding distance.
- *Translations*: for fixed $v \in \mathbb{R}^n$,

$$t_v(x) = x + v$$

  is an isometry, called *translation by* v.

- t_v isometry whenever $d(x, y) = |y - x|$ for $|x|$ a *norm*

$$d(t_v(x), t_v(y)) = |(y + v) - (x + v)| = |y - x| = d(x, y)$$

- Let $\mathcal{T} = \{t_v \mid v \in \mathbb{R}^n\}$
- $\mathcal{T}$ is a *subgroup* of $\mathcal{I}$
- Why?
- Let $\mathcal{I}_0 = \{g \in \mathcal{I} \mid g(0) = 0\}$
- $\mathcal{I}_0$ is a *subgroup* of $\mathcal{I}$.

$e \in \mathcal{I}$

$\mathcal{I}$ closed under

$\circ\,,\ {}^{-1}$

$f, g \in \mathcal{I} \Rightarrow f \circ g \in \mathcal{I}$

$f \in \mathcal{I} \Rightarrow f^{-1} \in \mathcal{I}$

$t_w \circ (t_v) = t_{(w+v)}$

$t_v^{-1} = t_{(-v)}$

$\mathbf{0} = (0,0)$

t/w
taxicab

$e \in \mathcal{G}_0$

closed under $\circ$, $^{-1}$

$f, g \in \mathcal{G}_0$

$f(0) = 0, \; g(0) = 0$

$f(\underbrace{g(0)}_{0}) = f(0) = 0$

- *Question*: Normality?

g_0

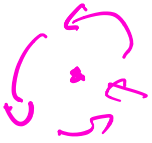
$t_\sigma g_0 t_\sigma^{-1} = g_{t_\sigma(0)}$

$H \subset G$

H normal if $gHg^{-1} \subset H$

$\forall h \in H,\ ghg^{-1} \in H$

$\forall g \in G$

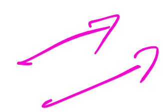


choose $t_\sigma g t_\sigma^{-1}$

0

Not normal: choose something to define gp

Normals define it indep of choice



$\mathcal{J}$ normal

$$\mathcal{J} = \mathcal{J}(\mathbb{R}^2, d_{Euclidean})$$

$$d_{(2)}$$

- If $f \in \mathcal{I}$, then

$$f = \underbrace{t_v \circ g}$$

where $g \in \mathcal{I}_0$.

$f(x) = (t_v \circ g)(x)$

$= t_v(g(x))$

$= g(x) + v$

$f(x) = g(x) + v$

for g fixing 0; $g(0) = 0$

- In fact,

$$v = f(0) \quad \text{and} \quad g = t_{-f(0)} \circ f$$

f

0 $\quad$ $f(0)$ $\quad$ $t^{-1}_{f(0)} \circ f^{-1}(0)$

$(t^{-1}_{f(0)} f(0))$

$= 0$

- Explicitly,

$$f(x) = g(x) + f(0)$$

where

$$g(x) = f(x) - f(0)$$

$f(x) = (f(x - f(0)) + f(0).$

- *Conclusion*: to know $\mathcal{I}$, sufficient ot know $\mathcal{I}_0$.
- Homework: for $d_{(1)}$ = taxicab,

  $$\mathcal{I}_0 = \{(x_1, x_2) \to (\pm x_1, \pm x_2)\} \cup \{(x_1, x_2) \to (\pm x_2, \pm x_1)\}$$

  (eight elements)
- *Question*: $\mathcal{I}_0$ for $d_{(2)}$ = Euclidean metric.

Rotations

$d = d_{Eucl}$

Theorem

If $g \in \mathcal{I}_0(\mathbb{R}^n, d_{(2)})$, then g is linear.

Pf $\quad g : \mathbb{R}^2 \to \mathbb{R}^2$ isometry

$g(0) = 0$

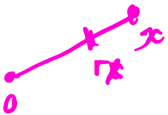
to prove:

$g(rx) = rg(x)$ $\forall r \in \mathbb{R}$

$g(x+y) = g(x) + g(y)$ $\forall x, y \in \mathbb{R}^2$

$0 < r < 1$

$g(rx)$

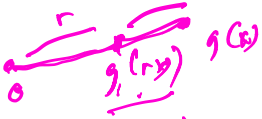


$rx \in$ line $0x \Leftrightarrow d(0,x) = d(0, rx) + d(rx, x)$

g isom

$$d(\underbrace{g(0)}_{\text{is } 0}, g(x)) = d(g(0), g(rx)) + d(g(rx), g(x))$$

$$d(0, g(x)) = d(0, g(rx)) + d(g(rx), g(x))$$

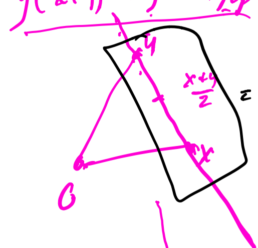
$\Rightarrow$ ① 

$$d(0, g(rx)) = \underbrace{d(0, rx)}_{r\, d(0,x)}$$

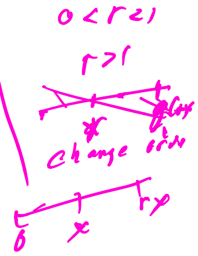
$$d(0, r\,g(x)) = d(0, rx)$$

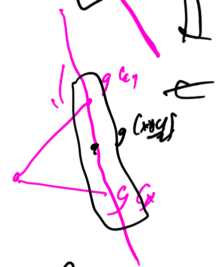
$$\Rightarrow \boxed{g(rx) = r\,g(x)}$$

$g(x+y) = g(x) + g(y)$



= in Dm





= in D

$g\left(\frac{x+y}{2}\right) =$ on segment $g(x)$ $g(y)$ half way

$$g\left(\frac{x+y}{2}\right) = \frac{g(x)+g(y)}{2}$$

$$2\left(g\left(\frac{x}{2}\right) + g\left(\frac{y}{2}\right)\right) = \frac{g(x)}{2} + \frac{g(y)}{2}$$

$$g\left(2\left[\frac{x+y}{2}\right]\right) \quad g(x)+g(y)$$

$$g(x+y) = g(x)+g(y)$$

$g(x) = Ax$ 2×2 matrix A

$$g\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$|\det(A)| = 1$

$$|Ax|^2 = |x|^2$$

$(x_1\ x_2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$x^t x$

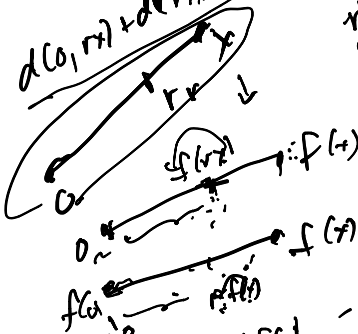
$= x_1^2 + x_2^2$

$$|Ax|^2 = (Ax)^t (Ax) = x^t \underbrace{\boxed{A^t A}}_{=I} x = x^t x$$

orthogonal $\boxed{A^t A = I}$

0 < r < 1

$d(0, rx) + d(rx, y) = d(0, x)$



$r\|x\|$

$f(rx) = r f(x)$

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad A^{-1} = A^{t}$$

$a_{11}^2 + a_{21}^2 = 1 \quad a_{11} a_{12} + a_{21} a_{22} = 0$

Corollary

If $g \in \mathcal{I}_0(\mathbb{R}^2, d_{(2)})$, then either

$$g\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

or

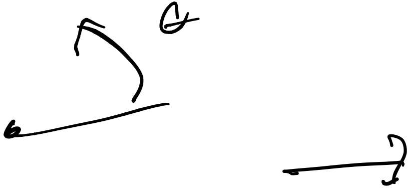
$$g\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

For $0 \leq \theta \leq 2\pi$.

Cx numbers $\mathbb{R}^2 \longleftrightarrow \mathbb{C}$

$$z \to \underline{e^{i\theta} z} = (\cos\theta + i\sin\theta)(x+iy)$$

$$z \to e^{i\theta} \bar{z} \quad (\qquad\qquad) (x - iy)$$



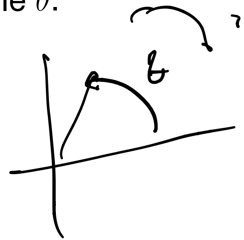
Geometric Interpretation

$\det = \cos^2\theta - (-\sin^2\theta)$

$= 1$

- $$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

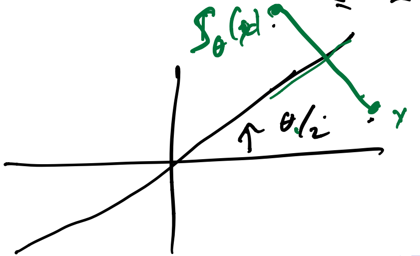
  has determinant -1, represents counterclockwise rotation by angle θ.

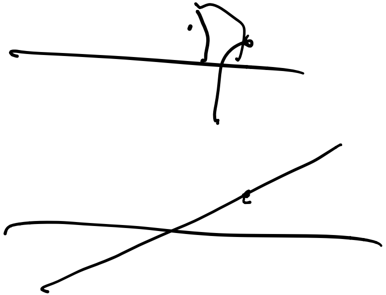


- $$S_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

  has determinant $= -1$, represents reflection in the line $\{t(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}) \mid t \in \mathbb{R}\}$

$$\det = -\cos^2\theta - \sin^2\theta$$
$$= -1$$





Euclidean Isometries of $\mathbb{R}^2$

$f^{+}_{\theta,v} : \mathbb{R}^2 \to \mathbb{R}^2$

$f^{-}_{\theta,v} : \mathbb{R}^2 \to \mathbb{R}^2$

- Let $v \in \mathbb{R}^2$ and let $\theta \in [0, 2\pi]$.
- Let
$$f^{+}_{\theta,v}(x) = \boxed{R_\theta(x) + v}$$
- Let
$$f^{-}_{\theta,v}(x) = \boxed{S_\theta(x) + v}$$
- Let
$$\mathcal{I}^{\pm} = \{f^{\pm}_{\theta,v} \mid v \in \mathbb{R}^2,\ \theta \in [0, 2\pi]\}$$

orientation preserving

orient reversing

Theorem

det 1

det −1

1. $\mathcal{I} = \mathcal{I}^+ \cup \mathcal{I}^-$.
2. *In words, if $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a Euclidean isometry, then there exist $v \in \mathbb{R}^2$ and $\theta \in [0, 2\pi]$ so that either $f = f^+_{\theta,v}$ or $f = f^-_{\theta,v}$.*
3. *$\mathcal{I}^+$ is a subgroup of $\mathcal{I}$ of index two.*
4. $\det : \mathcal{I} \to \{\pm 1\}$ *is a surjective homomorphism with kernel $\mathcal{I}^+$.*

$\mathcal{I} \longrightarrow \{1, -1\}$ mult gp with 2 elements

Comments

$$f^{+}_{\theta,v}(x) = R_\theta(x) + v$$

- The correspondence $(\boxed{\theta}, v) \to f^+_{\theta,v}$ is bijective if either:
  - We take $\boxed{\theta} \in [0, 2\pi)$: un-natural.
  - We take $\theta \in$ circle S^1 : more natural, will do later.
- Similar statements for $f^-_{\theta,v}$.
- Formulas for

$$f^+_{v_1,\theta_1} \circ f^+_{v_2,\theta_2}$$

etc???

$$\mathcal{I}^+ \longleftrightarrow \left\{ (\theta, v) : \begin{array}{l} \theta \in S^1 \\ v \in \mathbb{R}^2 \end{array} \right\}$$

$$\Leftarrow S^1 \times \mathbb{R}^2$$

$f = f^{+} = \det = 1$

$$\underline{f_{\theta_1, v_1}} \circ \underbrace{f_{\theta_2, v_2}(x)}$$

$$= R_{\theta_1}\left(R_{\theta_2}(x) + v_2\right) + v_1$$

$$= \underbrace{R_{\theta_1} R_{\theta_2}}(x) + R_{\theta_1} v_2 + v_1$$

$$= R_{\theta_1 + \theta_2}(x) + R_{\theta_1} v_2 + v_1$$

$$f_{\theta_1+\theta_2,\ \boxed{v_1 + R_{\theta_1} v_2}} \longrightarrow \text{not } v_1+v_2.$$

$$= f_{\theta_1, v_1} \circ f_{\theta_2, v_2}$$

$S^1 \times \mathbb{R}^2$

(θ, v)

$(\theta_1, v_1)(\theta_2, v_2)$

$= (\theta_1 + \theta_2,\ v_1 + R_{\theta_1} v_2)$

Fixed Points

$\mathcal{I}^+$ not abelian

$R_{\theta_2}(R_{\theta_1}(x) + v_1) + v_2$

$\boxed{\theta_2+\theta_1} \quad v_2 + R_{\theta_2} v_1$

$\neq \quad \boxed{\theta_1+\theta_2} \quad v_1 + R_{\theta_1} v_2$

$$\boxed{f_{\theta,\tau}(z) = e^{i\theta} z + \tau}$$

fixed pt: $f_{\theta,\tau}(z) = z$

$$e^{i\theta} z + \tau = z$$

$$\tau = z(1 - e^{i\theta})$$

$$z = \tau / 1 - e^{i\theta}$$

if. $e^{i\theta} \neq 1$

$\frac{v}{1-e^{i\theta}}$ = fixed pt

Rotation about the fixed pt.

$f_{\theta,v}$ has a fixed pt if $e^{i\theta} \neq 1$

$e^{i\theta} = 1$ $f(z) = z + v$ translation

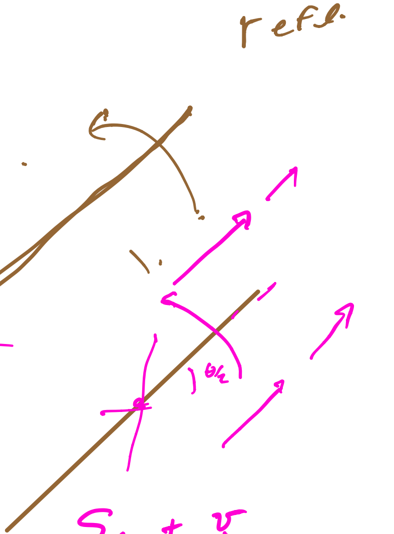
~~Composing rotations about different centers~~

f orient. preserving isom. of $\mathbb{R}^2$

1) a translation ⟵ no fixed pts

2) a rotation about P. ⟵ has a fixed pt, P.

fixed pts	refl, mirror
no fixed pts	glide refl



$S_\theta + \vec{v}_{\parallel}$ mirror

Extension next Wed 9/20
HW2

HW3 give Mon 9/18
due Mon 9/25 before midterm

Comments on HW 2:

$$E_d(x,y) = \{ z \in (X,d) : d(x,z) + d(z,y) = d(x,y) \}$$

$$f : (X,d) \xrightarrow{\text{say from}} (X',d')$$

$$f(E_d(x,y)) = E_{d'}(f(x), f(y))$$

$$\subset$$

$$\supset$$

$$\underline{f(E_d(x,y))} \subset E_{d'}(f(x), f(y))$$

$z \in E_d(x,y)$

$d(x,z) + d(z,y) = d(x,y)$

Want: $f(z) \in E_{d'}(f(x), f(y))$

$d'(f(x), f(z)) + d'(f(z), f(y))$

$= d'(f(x), f(y))$

f isometry $\Rightarrow d'(f(x), f(y)) = d(x,y)$

$d'(f(x), f(z)) = d(x,z)$

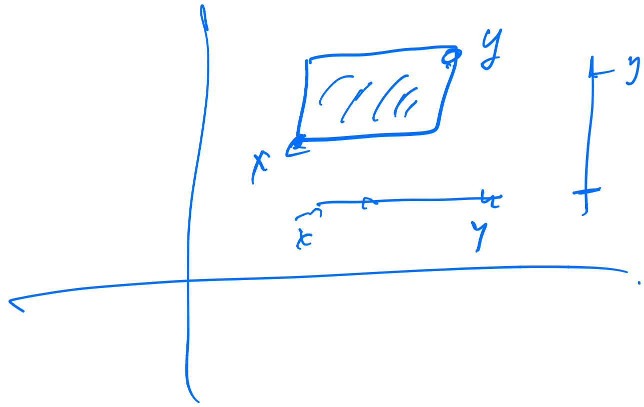
$d'(f(z), f(y)) = d(z,y)$

$d(x,z) + d(z,y) = d(x,y)$

$\Downarrow$

$d'(f(x), f(z)) + d'(f(z), f(y)) = d'(f(x), f(y))$

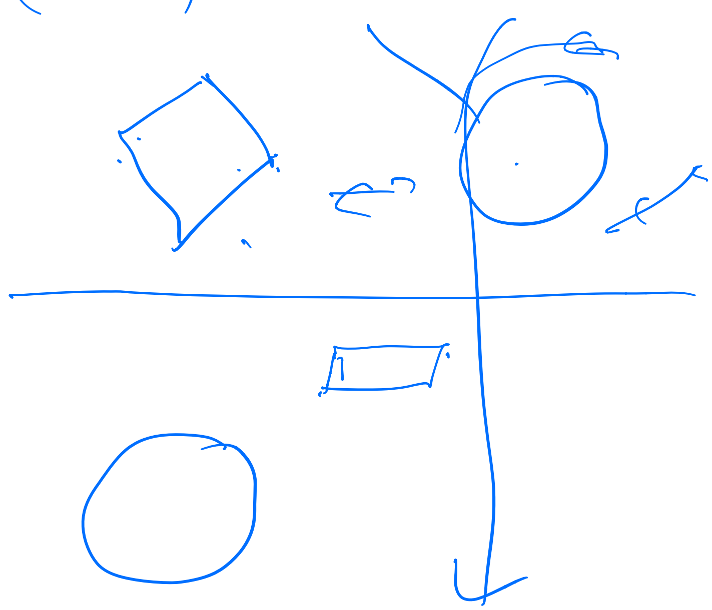
$\Rightarrow f(z) \in E_{d'}(f(x), f(y))$



Isometry?

How to tell?

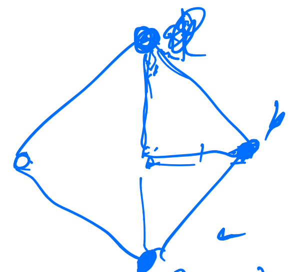
$(\mathbb{R}^2, \text{taxicab})$ $(\mathbb{R}^2, \text{Eucl})$

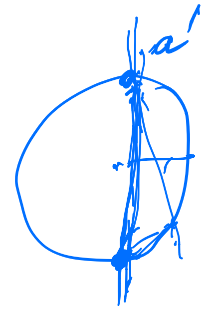


not isom

Unit spheres (subspace metric)

not isom





$x, y \in S(0,1)$

$d(x,y) \le 2$

$\le d(x,0) + d(y,0)$

$1 + 1$

$d(a,b) = 2$

$d(a,c) = 2$

$a \to b$, $a \to c$

$d(a,b) = 2$

$d(a,c) = 2$

$b \ne c$.

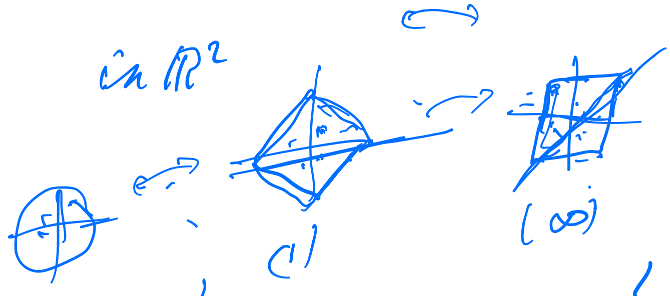
given $p \in S_E(0,1)$

$\exists ! \ q \in S_E(0,1)$

$d(p,q) = 2$

$d_{(1)}$ & $d_{(\infty)}$ isometric

in $\mathbb{R}^2$



(1)

(∞)

Prove $(\mathbb{R}^2, d_{(1)})$

& $(\mathbb{R}^2, d_{(\infty)})$

are isometric

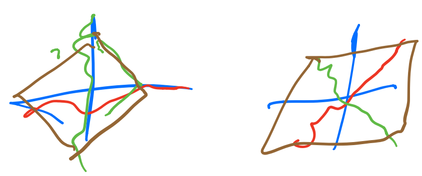
(x_1, x_2)

$(x_1, 0) \longrightarrow (x_1, x_1)$

$(x_1, x_2) \longrightarrow (x_1 + x_2, x_1 - x_2).$

$(x_1, 0) \longrightarrow (x_1, x_1)$

$(0, x_2) \longrightarrow (x_2, -x_2)$



Extra credit problem:

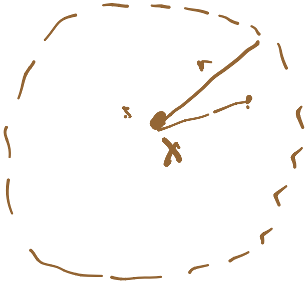
are $(\mathbb{R}^3, d_{(1)})$ and $(\mathbb{R}^3, d_\infty)$ isometric ??

Solution accepted any time during the semester

Topology of Metric Spaces

defined later

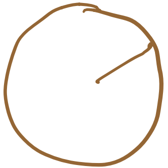
- (X, d) metric space, $x \in X$, $r \geq 0$. Define
  1. $B(x, r) = \{y \in X : d(x, y) < r\}$, the *ball of radius r centered at x*.
  2. $\bar{B}(x, r) = \{y \in X : d(x, y) \leq r\}$, the *closed ball of radius r centered at x.* .
  3. $S(x, r) = \{y \in X : d(x, y) = r\}$, the *sphere of radius r centered at x.*



$r > 0$ $\quad B(x, r)$ $\quad \le$

$\{ y : d(x,y) < r \}$

"closed" ball



$\le$

Sphere

$$S(x,r) = \{ y : d(x,y) = r \}$$

Open sets

- Definition

$U \subset X$ *open set* $\iff \forall x \in U \;\; \exists r > 0$ so that

$$B(x, r) \subset U.$$

- Example: $(\mathbb{R}^n, d_{(2)})$ usual open sets.

$U \subset \mathbb{R}^n$ open

$\iff \forall x \in U \; \exists r > 0$ s.t. $B_{(2)}(x, r) \subset U$

(1)

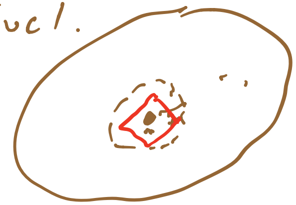
(∞)

- Example: $(\mathbb{R}^n, d_{(1)})$ or $(\mathbb{R}^n, d_{(\infty)})$: same open sets.
- Proof?

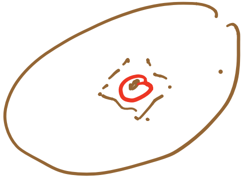
U open (Eucl)

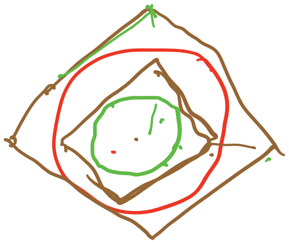
open (Tax)

U open Eucl.



U. op in Tax





$$B_T(x,1) \subset B_E(x,1) \subset B_T(x,\sqrt{2})$$

$$\underline{B_T(x,r) \subset U} \Rightarrow B_E\left(x, \frac{r}{\sqrt{2}}\right) \subset U$$

- (X, d) discrete metric space $\Longrightarrow$ all sets are open.

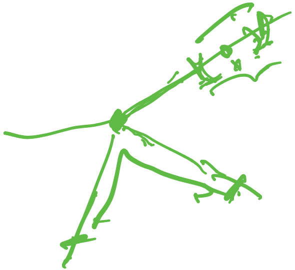
$$d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

for $A \subset X$ any subset

$x \in A$ $B(x, \tfrac{1}{2}) \subset A$?

$\underbrace{B(x, \tfrac{1}{2})}_{\{x\}} \subset A$

- Open sets in French railway metric.



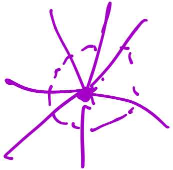
$B(x, r)$

$r < \underline{|x|}$

2

- ~~Examples on non-open sets:~~

$B(0,r)$



$\forall\, x \neq 0,$

$B(0,r) =$ ① interval about x in line $0x$ if $r < |x|$

② interval $\cup$ Euclidean ball of radius r about 0.

- Theorem
  (X, d) *metric space,* $x \in X$ *and* $r > 0 \implies$

  $$B(x, r) \text{ is an open set}$$

  - Proof?

- Theorem
  (X, d) *metric space,* $x \in X$ *and* $r \geq 0 \implies$

  $$\{y \in X | d(x, y) > r\} \textit{ is an open set .}$$

  - Proof?