Introduction to Algebraic and Geometric Topology Week 14

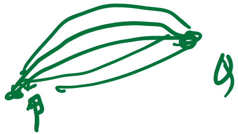
Domingo Toledo

University of Utah

Fall 2017

Recall: First Variation Formula for Arc-Length

- $S \subset \mathbb{R}^3$ a smooth surface (given by $f = 0, \nabla F \neq 0$)
- $\gamma : [0, L_0] \to S$ a smooth curve, parametrized by arclength, of length L_0
- endpoints $P = \gamma(0)$ and $Q = \gamma(L_0)$.
- Want necessary condition for γ to be shortest smooth curve on S from P to Q
- Calculus: consider *variations of* γ.
- This means: a "curve $c(t)$ of curves" with $c(0) = \gamma$.
- More precisely: a smooth map $\tilde{\gamma}$



$\tilde{\gamma} : [0, L_0] \times (-\epsilon, \epsilon) \to S$ with $\tilde{\gamma}(s, 0) = \gamma(s)$ for all $s \in [0, L_0]$.

with s being arclength on $\tilde{\gamma}(s, 0)$ but not necessarily on $\tilde{\gamma}(s, t)$ for $t \neq 0$.

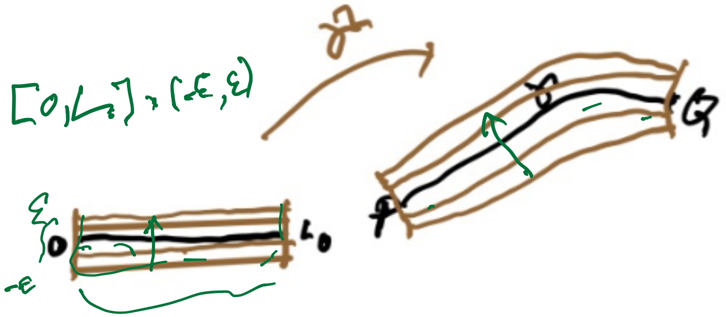


Figure: A Variation of γ

- If, in addition, we have that

$$\tilde{\gamma}(0, t) = P, \; \tilde{\gamma}(L_0, t) = Q \text{ for all } t \in (-\epsilon, \epsilon),$$

we say that $\tilde{\gamma}$ is a *variation of γ with fixed endpoints*.

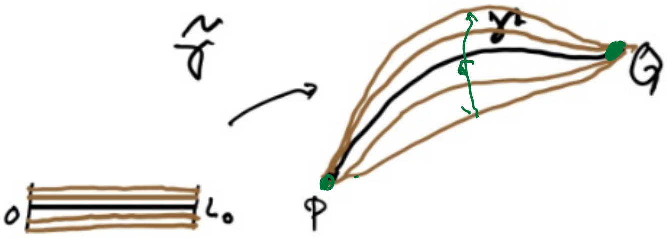


Figure: Variation of γ with Fixed Endpoints

$\tilde{\gamma}(s, t)$

$\tilde{\gamma}(s, 0) = \gamma(s)$

- Let

$$\boxed{L(t) = \int_0^{L_0} \left|\frac{\partial \tilde{\gamma}}{\partial s}(s, t)\right| \, ds.}$$

$\left|\frac{\partial \tilde{\gamma}}{\partial s}(s, 0)\right| = 1$

- Necessay condition for a minimum:

$$\frac{dL}{dt}(0) = 0$$

for all variations $\tilde{\gamma}$ of γ with fixed endpoints P, Q.

- Let's compute $\frac{dL}{dt}(0)$ for arbitrary variations, then specialize to variations with fixed endpoints.

- Begin with the formula for $L(t)$

$$L(t) = \int_0^{L_0} (\tilde{\gamma}_s(s,t) \cdot \tilde{\gamma}_s(s,t))^{1/2}\, ds$$

- Differentiate under the integral sign

$$\frac{dL}{dt} = \int_0^{L_0} \frac{1}{2}(\tilde{\gamma}_s(s,t)\cdot\tilde{\gamma}_s(s,t))^{-1/2}(2\,\tilde{\gamma}_{st}(s,t)\cdot\tilde{\gamma}_s(s,t))\, ds.$$

at t=0

- Evaluate at $t = 0$ using that $\tilde{\gamma}_s(s,0) \cdot \tilde{\gamma}_s(s,0) = 1$

$$\frac{dL}{dt}(0) = \int_0^{L_0} \tilde{\gamma}_{st}(s,0) \cdot \tilde{\gamma}_s(s,0)\, ds.$$

$\frac{\partial}{\partial t}\left[\frac{\partial}{\partial s}\right]$

- Equality of mixed partial derivatives gives

$$\frac{dL}{dt}(0) = \int_0^{L_0} \tilde{\gamma}_{ts}(s,0) \cdot \tilde{\gamma}_s(s,0)\, ds.$$

- Integrate by parts, using the formula

$$(\tilde{\gamma}_t(s,0)\cdot\tilde{\gamma}_s(s,0))_s = \tilde{\gamma}_{ts}(s,0)\cdot\tilde{\gamma}_s(s,0) + \tilde{\gamma}_t(s,0)\cdot\tilde{\gamma}_{ss}(s,0)$$

- Get

$$\frac{dL}{dt}(0) = (\tilde{\gamma}_t(s,0)\cdot\tilde{\gamma}_s(s,0))\big|_0^{L_0} - \int_0^{L_0} \tilde{\gamma}_t(s,0)\cdot\tilde{\gamma}_{ss}(s,0)\, ds$$

- Define a vector field $V(s)$ along γ by

$$V(s) = \tilde{\gamma}_t(s, 0).$$

- This is called the *variation vector field*.
- $V(s)$ is the velocity vector of the curve $t \to \tilde{\gamma}(s, t)$ at $t = 0$.
- $V(s)$ tells us the velocity at which $\gamma(s)$ initially moves under the variation.
- If the variation preserves endpoints, then $V(0) = 0$ and $V(L_0) = 0$,

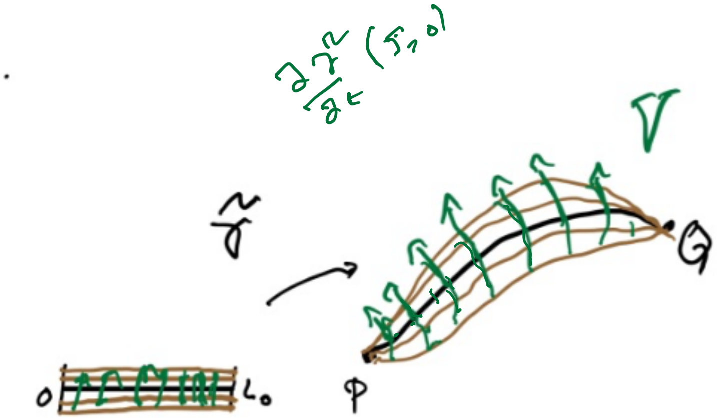


Figure: Variation Vector Field

First Variation Formula:

- Earlier

$$\frac{dL}{dt}(0) = \cancel{V(s) \cdot \gamma'(s)|_0^{L_0}} - \int_0^{L_0} \overline{V(s)} \cdot \gamma''(s)^T ds.$$

- Since $V(s)$ is tangent to S, we replaced $\gamma''(s)$ by its tangential component γ''^T
- Necessary condition for minimum: $\frac{dL}{dt}(0) = 0$ for all variations $\tilde{\gamma}$ of γ with fixed endpoints.
- equivalently

$$\boxed{\int_0^{L_0} V(s) \cdot \gamma''^T} = 0 \; \forall \; V \text{ along } \gamma \text{ with } v(0) = V(L_0) = 0$$

- Finally this means $\gamma''^T \equiv 0$.
- Reason: use “bump functions”

choose $V(s) = \varphi(s)(\gamma'')^T$

$\int \varphi(s)\,[(\gamma'')^T]^2 = 0$

$\forall \varphi$

$\Rightarrow (\gamma'')^T(s_0) = 0$

$\forall \varphi$ $\int \varphi(s)\,ds = 1$,

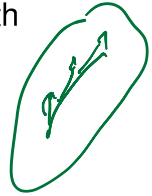
$\gamma''^T(s_0) = 0$

Covariant Derivative and Geodesic Equation

$S \subset \mathbb{R}^3$

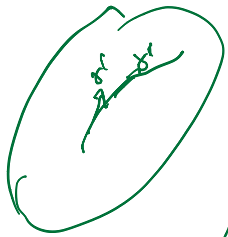
Definition

Let $\gamma : (a, b) \to S$ be a smooth curve and $V : (a, b) \to \mathbb{R}^3$ a smooth vector field along γ, meaning that V is a smooth map and for all $s \in (a, b)$, $V(s) \in T_{\gamma(s)}S$, the tangent plane to S at $\gamma(s)$.



1. The tangential component $V'(s)^T$ is called the *covariant derivative* of V and is denoted DV/Ds.
2. γ is a *geodesic* if and only if $D\gamma'/Ds = 0$ for all $s \in (a, b)$.
3. $\kappa_g(s) = |D\gamma'/Ds|$ is called the *geodesic curvature* of γ.

Intrinsic Geometry



$$\boxed{\frac{D\gamma'}{ds} \equiv 0}$$

Geodesic

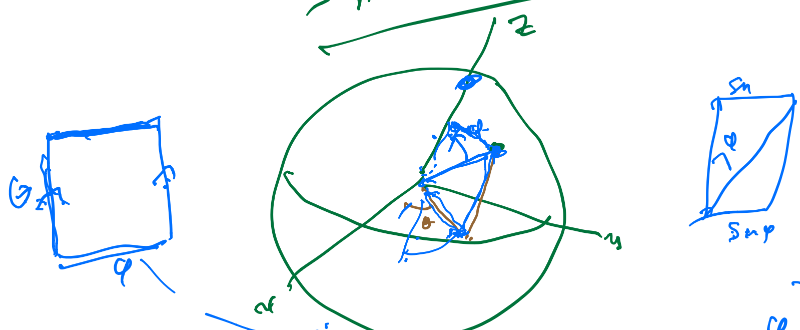
geodesic eq

$$\frac{D\gamma'}{ds} = \left(\gamma''\right)^T$$

Nec Cond for min geodesic

$$\Rightarrow \frac{D\gamma'}{ds} \equiv 0 \xLeftarrow{\text{def}}$$

Spherical coords



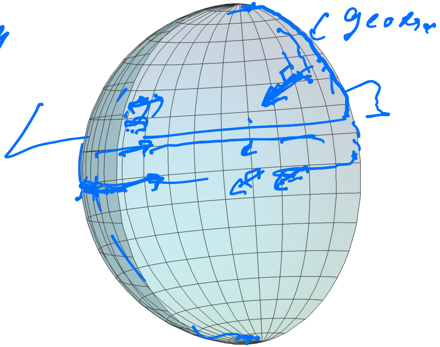
$$\vec{\gamma}(\varphi,\theta) = \left(\sin\varphi\cos\theta,\ \sin\varphi\sin\theta,\ \cos\varphi\right)$$

$$0 \le \varphi \le \pi, \quad 0 \le \theta \le 2\pi$$

Geodesics on S^2

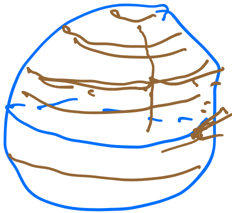
φ = co-latitude

θ = longitude



φ = const

- Equator and meridans are geodesics.
- Equator a local maximum among the parallels.

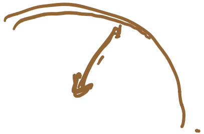


$L(\varphi = \text{const}) = 2\pi \sin\varphi$

local max

max at $\pi/2$

back to Calc III



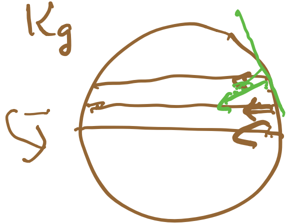
$\gamma(s)$ $s = \text{arclength}$

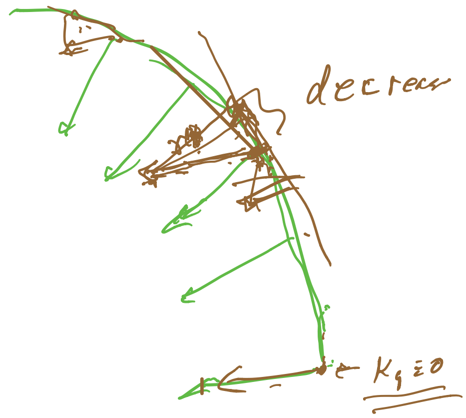
$|\gamma''(s)| = \text{curvature} = K(s)$

$$\left| \frac{D\gamma'}{Ds} \right| = K_g(s) = \text{geodesic curv}$$

γ geod $\iff K_g \equiv 0$

parallels on S^2, const K_g





decrease
$K_g = 0$

- If V, W are vector fields along γ, then

$$\frac{d}{ds}(V \cdot W) = \frac{DV}{Ds} \cdot W + V \cdot \frac{DW}{Ds}.$$

- If γ is parametrized by arc-length, $\gamma' \cdot \gamma' \equiv 1$
- Therefore

$$\frac{D\gamma'}{Ds} \cdot \gamma' \equiv 0$$

- Thus $\frac{D\gamma'}{Ds}$ is tangent to S and normal to γ'.

- The first variation formula says that, if the endpoints are fixed, length decreases most rapidly if we move in the direction of $\frac{D\gamma'}{Ds}$.
- This means: if the variation field $V(s) = \phi(s)\frac{D\gamma'}{Ds}(s)$ for $\phi(s) > 0$ on $(0, L_0)$:

$\varphi V(s) \cdot \frac{D\gamma'}{Ds}$

$$\frac{dL}{dt}(0) = -\int_0^{L_0} \phi(s)\left|\frac{D\gamma'}{Ds}(s)\right|^2 \, ds$$

- Check this with parallels in S^2.

$V(s) = \varphi(s) \frac{D\gamma'}{Ds}$

Length of curves in charts

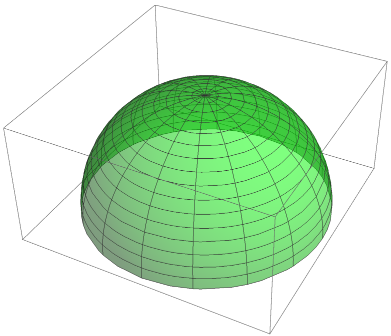
- Recall smooth surface $S = \{f(x, y, z) = 0\} \subset \mathbb{R}^3$,
- $\nabla f \neq 0$ on S,
- Chart (U, ϕ) on S:

$$\begin{array}{ccccc} U & \subset & S & \subset & \mathbb{R}^3 \\ \phi \big\downarrow & & & & \\ V & & \subset & & \mathbb{R}^2 \end{array}$$

- $\phi^{-1}(u, v) = \mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v))$.
- $f(x(u, v), y(u, v), z(u, v)) \equiv 0$.
- $\mathbf{x} : V \to S$ is a "parametrization" of $U \subset S$.

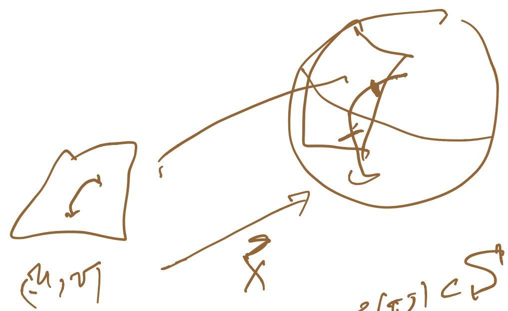
Example

- $f(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$ sphere.
- $U = \{(x, y, z) \in S \mid z > 0\}$ upper hemisphere,
- $V = \{(u, v) \in \mathbb{R}^2 \mid u^2 + v^2 < 1\}$ unit disk in $\mathbb{R}^2$.
- $\phi(x, y, z) = (x, y)$ projection.
- $\mathbf{x}(u, v) = (u, v, \sqrt{1 - u^2 - v^2})$.
- $\mathbf{x}$ is a parametrization of the upper hemisphere.



- Suppose $\gamma : [a, b] \to S$ lies in coordinate chart (U, ϕ).
- Then $\gamma(t) = \mathbf{x}(u(t), v(t))$ for a unique curve

  $(u(t), v(t))$ lying in V, namely $\phi \circ \gamma(t) = (u(t), v(t))$.
- It is often convenient to do the calculations in terms of $(u(t), v(t))$.
- Start with $\gamma(t) = \mathbf{x}(u(t), v(t))$.
- $\gamma'(t) = \mathbf{x}_u u'(t) + \mathbf{x}_v v'(t)$ is its tangent vector.
- $\gamma'(t) \cdot \gamma'(t)$ is the norm squared of γ'



$\vec{x} : U \xrightarrow{\simeq} \vec{x}(U) \subset S$ chart

γ in S

$(u(t), v(t)) \longrightarrow \gamma(t)$

e.g., $u, v = (\varphi, \theta)$

$\vec{x}(\varphi, \theta) = (\sin\varphi \cos\theta, \dots,)$

$\vec{x}(\varphi(t), \theta(t))$ curve on S^2

Length of $\vec{x}(u(t), v(t))$

$$\left| \frac{d\vec{x}}{dt} \right|^2$$

Chain rule: $\frac{d\vec{x}}{dt} = \vec{x}_u u' + \vec{x}_v v'$

$$\frac{d\vec{x}}{dt} \cdot \frac{d\vec{x}}{dt} = (\vec{x}_u u' + \vec{x}_v v') \cdot (\vec{x}_u u' + \vec{x}_v v')$$

$$= \underbrace{(\vec{x}_u \cdot \vec{x}_u)}(u')^2 + 2(\vec{x}_u \cdot \vec{x}_v) u' v' + (\vec{x}_v \cdot \vec{x}_v) v'^2$$

alternately;

$$\boxed{ds^2 = \underbrace{(\vec{x}_u \cdot \vec{x}_u)}\,du^2 + 2\,\underbrace{(\vec{x}_u \cdot \vec{x}_v)}\,du\,dv + \underbrace{(\vec{x}_v \cdot \vec{x}_v)}\,dv^2}$$

get ~~the~~ the length s by integrating

- Explicitly $\gamma'(t) \cdot \gamma'(t) =$

  $(\mathbf{x}_u u' + \mathbf{x}_v v') \cdot (\mathbf{x}_u u' + \mathbf{x}_v v')$

  $= (\mathbf{x}_u \cdot \mathbf{x}_u) u'^2 + 2(\mathbf{x}_u \cdot \mathbf{x}_v) u'v' + (\mathbf{x}_v \cdot \mathbf{x}_v) v'^2$

- At this point it is best to forget the curve γ altogether, work only with the expression

  $$ds^2 = (\mathbf{x}_u \cdot \mathbf{x}_v) du^2 + 2(\mathbf{x}_u \cdot \mathbf{x}_v) du dv + (\mathbf{x}_v \cdot \mathbf{x}_v) dv^2$$

- What does this mean?

- The length of a curve is given by integrating the length of its tangent vector:

$$L(\gamma) = \int_a^b (\gamma'(t) \cdot \gamma'(t))^{\frac{1}{2}} dt$$

- Could equally well be written as

$$L(\gamma) = \int_\gamma ds$$

- What is ds?

- $d\mathbf{x} : T_{(u,v)}V \to T_{\mathbf{x}(u,v)}\mathbb{R}^3$.
- If $\mathbf{v} \in T_{(u,v)}V$, $ds(\mathbf{v}) = |d\mathbf{x}(\mathbf{v})| = (d\mathbf{x}(\mathbf{v}) \cdot d\mathbf{x}(\mathbf{v}))^{\frac{1}{2}}$
- ds is a function of two (vector) variables,(equivalently 4 real variables):
    1. a point $p = (u, v) \in V$.
    2. a tangent vector $\mathbf{v} = (u', v') \in T_pV$

- Notation can be confusing: u, v, u', v' can mean
  1. Independent variables. Then $ds_{(u,v)}(u'.v') = |d_p\mathbf{x}(\mathbf{v})|$ as above.
  2. If evaluated on a curve $(u(t), v(t)), a < t < b$, then ds means the function of one variable

$$|d_{(u(t),v(t))}\mathbf{x}(u'(t), v'(t))|$$

     where

$$u'(t) = \frac{du}{dt}, \quad v'(t) = \frac{dv}{dt}$$

- Going back to $\mathbf{x} : V \to U \subset S \subset \mathbb{R}^3$, and $(p, \mathbf{v}) \in T_p V$,
- ds^2 is a function of $p, \mathbf{v}$, quadratic in $\mathbf{v}$.
- $d_p s(\mathbf{v})^2 = |d_p \mathbf{x}(\mathbf{v})|^2$ (usual square norm in $\mathbb{R}^3$).
- $d_p s^2(\mathbf{v}) = (\mathbf{x}_u \cdot \mathbf{x}_u) du^2 + 2(\mathbf{x}_u \cdot \mathbf{x}_v) du dv + (\mathbf{x}_v \cdot \mathbf{x}_v) dv^2$,
- du, dv are functions of $p, \mathbf{v}$, linear in $\mathbf{v}$
- If $\mathbf{v} = (u', v'), d_p u(\mathbf{v}) = u',\ d_p v(\mathbf{v}) = v'$

- In summary, we get

$$ds^2 = g_{11}du^2 + 2g_{12}dudv + g_{22}dv^2$$

where g_{11}, g_{12}, g_{22} are smooth functions of u, v

- Moreover, at every $(u, v) \in V$, the matrix

$$G = \begin{pmatrix} g_{11}(u, v) & g_{12}(u, v) \\ g_{21}(u, v) & g_{22}(u, v) \end{pmatrix}$$

is symmetric and positive definite.

- In fact,

$$\begin{pmatrix} u' & v' \end{pmatrix} \begin{pmatrix} g_{11}(u,v) & g_{12}(u,v) \\ g_{21}(u,v) & g_{22}(u,v) \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix}$$

is the same as

$$\gamma'(t) \cdot \gamma'(t) = (\mathbf{x}_u u' + \mathbf{x}_v v') \cdot (\mathbf{x}_u u' + \mathbf{x}_v v')$$

which is ≥ 0, and, since $\mathbf{x}_u, \mathbf{x}_v$ are linearly independent, $= 0$ if and only if $(u', v') = (0, 0)$

- Equivalent Statement

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{x}_u \cdot \mathbf{x}_u & \mathbf{x}_u \cdot \mathbf{x}_v \\ \mathbf{x}_v \cdot \mathbf{x}_u & \mathbf{x}_v \cdot \mathbf{x}_v \end{pmatrix}$$

- We see again that G is symmetric and positive definite.

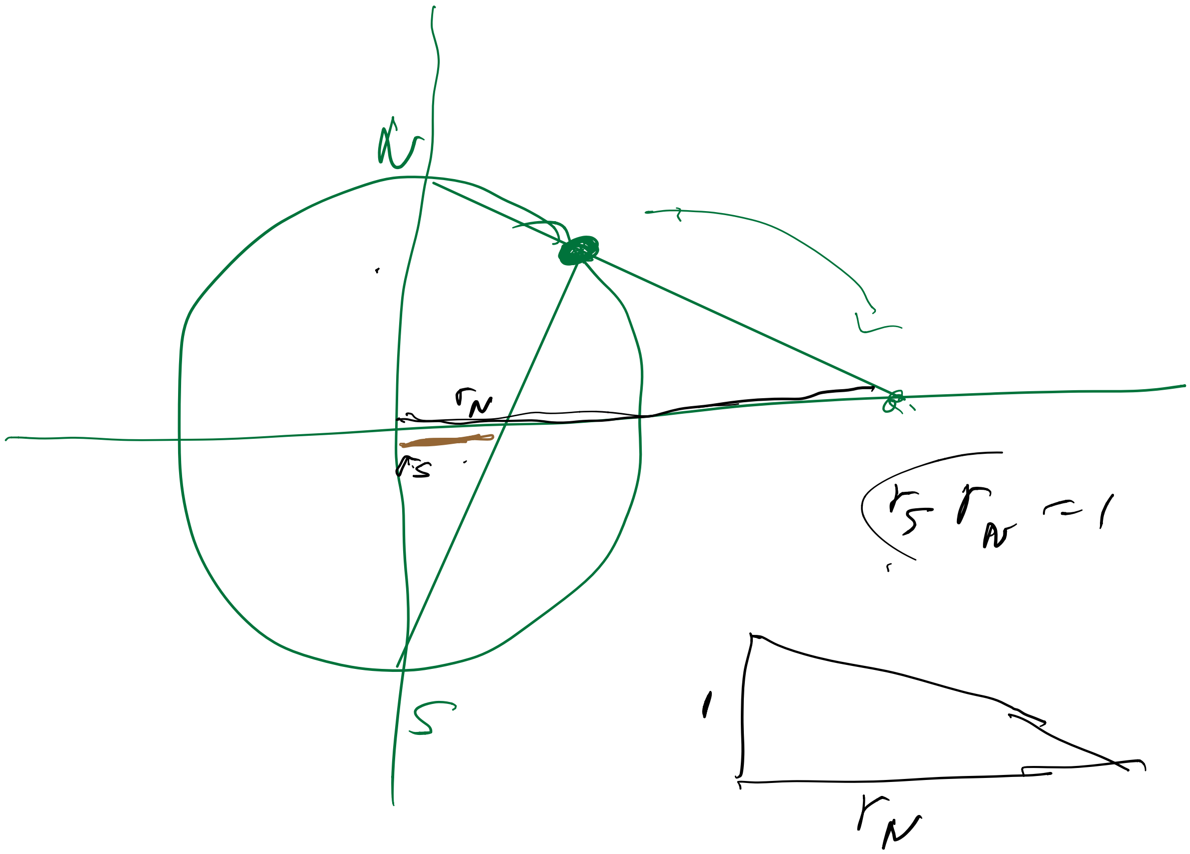
- Going back to $\gamma : [a, b] \to U \subset S$, piecewise smooth,

  $\gamma(t) = \mathbf{x}(u(t), v(t))$,
- Recall that the length of γ is

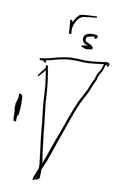
  $$L(\gamma) = \int_a^b ds = \int_a^b (g_{11}(u')^2 + 2g_{12}(u'v') + g_{22}(v')^2)^{\frac{1}{2}} dt$$

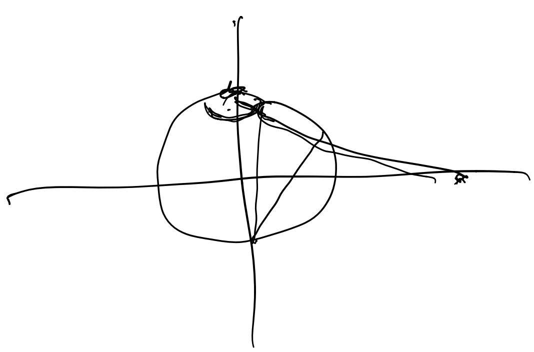
  where the g_{ij} are evaluated at $(u(t), v(t))$
- Usually work with the expression for ds^2 without using γ explictly.

Some Remarks on Homework



$r_S \, r_N = 1$

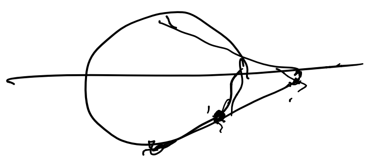




$\mathbb{R}^2$

$$f(x,y) = g\left(\frac{x,y}{x^2+y^2}\right)$$ g a smooth func in a nbhd of 0

$1, x, \ldots$



$f(x) = g(1/x)$ for g sm...

$x \quad 0 \quad 1/x$ $\qquad x \to 1/x$

$1/x = \frac{1}{1/x} = x$

x

$1/x$ extends

---

f extends to a smooth func at 0

$f\left(\frac{x,y}{x^2+y^2}\right)$ extends at 0

$x \to \frac{x}{x^2+y^2} \qquad \frac{1}{x^2+y^2}$

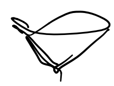
$$= \frac{1}{\left(\frac{x}{x^2+y^2}\right)^2 + \left(\frac{y}{x^2+y^2}\right)^2}$$

$$-\quad \frac{(x^2+y^2)^2}{x^2+y^2} = x^2+y^2$$ smooth at $(0,0)$

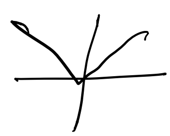
$$\frac{1}{\sqrt{x^2+y^2}} \qquad \frac{1}{\sqrt{\left(\frac{x}{x^2+y^2}\right)^2+\left(\frac{y}{\sqrt{x^2+y^2}}\right)^2}}$$

$$= \frac{1}{\sqrt{\frac{x^2+y^2}{(x^2+y^2)^2}}} = \frac{x^2+y^2}{\sqrt{x^2+y^2}}$$

$$= \sqrt{x^2+y^2}$$



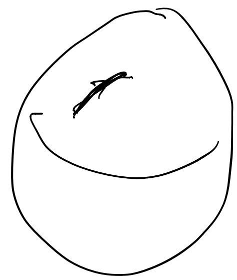
r



$f(z)$ at ∞

$\frac{1}{f\left(\frac{1}{z}\right)}$ at 0

Intrisic geom of surf



---

geodesic: curve $\gamma : (a,b) \to S$

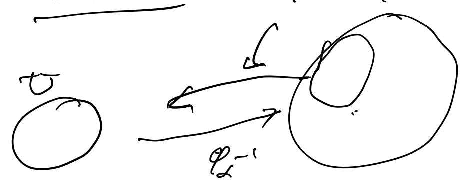
$$(\gamma'')^T \equiv 0$$

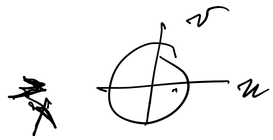
To study eq

"local coordinates"

"chart"

label: φ_α





$$\vec{x} : U \to \mathbb{R}^3$$

$$\searrow S^{\circ}$$

$$\gamma(t) = \boxed{\vec{x}\,(u(t), v(t))}$$

$$|\gamma'(t)|^2$$

$$|\vec{X}_u u' + \vec{X}_v v'|^2$$

$$= (\vec{X}_u u' + \vec{X}_v v') \cdot (\vec{X}_u u' + \vec{X}_v v')$$

$$= \underbrace{(\vec{X}_u \cdot \vec{X}_u)}_{E} u'^2 + 2\underbrace{(\vec{X}_u \cdot X_v)}_{F} u' v' + \underbrace{(\vec{X}_v \cdot X_v)}_{G} v'^2$$

E, F, G : Gauss (early 1800's)

g_{11}, g_{12}, g_{22} (Riemann, ----)

$$s = \int (\ )^{1/2}$$

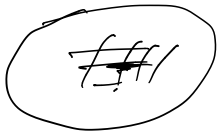
ds

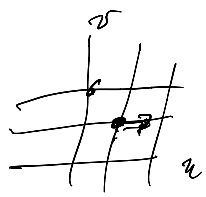
$$ds^2 = (X_u \cdot X_u)\, du^2 + 2(X_u \cdot X_v)\, du\, dv + (X_v \cdot X_v)(dv)^2$$

$$= g_{11}\, du^2 + 2g_{12}\, du\, dv + g_{22}(dv)^2$$

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$ 2x2 matrix of C^∞ fncs

Symm, positive definite





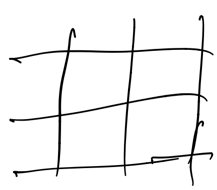
g_{11}, g_{12}, g_{22}

g_{11} = how x.. $(1,0)$

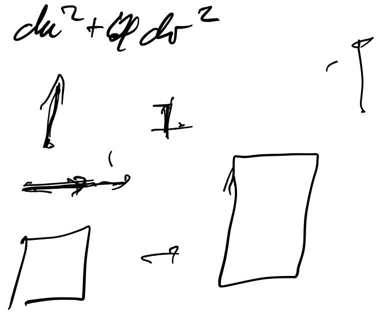
g_{22} -- $(0,1)^2$

g_{12} = dot prod $d\vec{x}(1,0) \cdot d\vec{x}(0,1)$

$(d\vec{x} \cdot d\vec{x})$



$du^2 + \cancel{G}\, dv^2$



Polar coords:

$\mathbb{R}^+ \times \mathbb{R} \xrightarrow{\vec{x}} \mathbb{R}^2$

$r, \theta \xrightarrow{\vec{x}} (r\cos\theta, r\sin\theta)$

$\cancel{dx/dr = (\cos\theta\, dr, \sin\theta\, dr)\, dr}$

$(dr\cos\theta - r\sin\theta\, d\theta,\ dr\sin\theta + r\cos\theta\, d\theta)$

$(dr\cos\theta - r\sin\theta\, d\theta)^2 + (dr\sin\theta + r\cos\theta\, d\theta)^2$

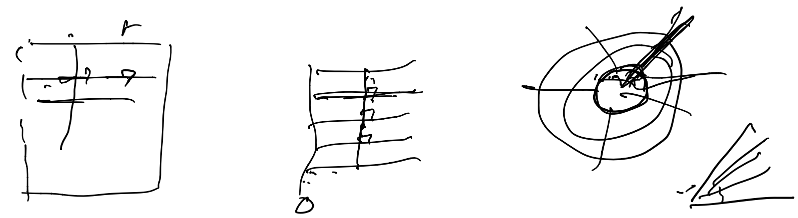
$dr^2\cos^2\theta - 2r\sin\theta\cos\theta\, dr\, d\theta + r^2\sin^2\theta\, d\theta^2$

$+ dr^2\sin^2\theta + 2r\cos\theta\sin\theta\, dr\, d\theta + r^2\cos^2\theta\, d\theta^2$

$dr^2 + 0 + r^2 d\theta^2$

$r \geq 0$

$= \boxed{dr^2 + r^2 d\theta^2}$



One Goal: Any surface,

any p,

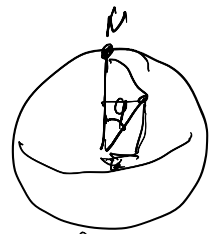
find "geodesic polar coordinates"

r, θ

$\boxed{dr^2 + g(r,\theta)^2 d\theta^2}$

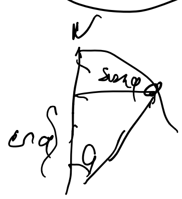
Ex: polar in plane.

Ex: Spherical Coords



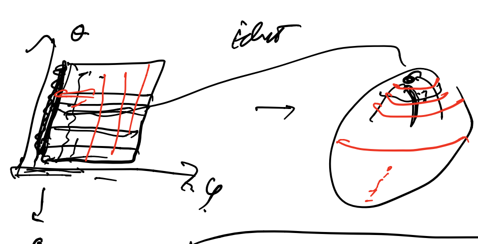
$\bar{X}(\varphi, \theta)$

$= (\sin\varphi \cos\theta, \sin\varphi \sin\theta, \cos\varphi)$



$d\bar{x} \cdot d\bar{x} = (\cdot \quad - - -$

$\varphi = 0 \to N$



$\boxed{d\varphi^2 + \sin^2\varphi \, d\theta^2}$

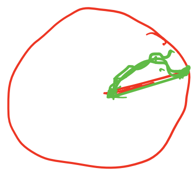
$\varphi = r$

Next: Thry of 2'nd order ODEs

$\to$ geodesic polar coords

$dr^2 + g(r,\theta)^2 d\theta^2$

$g(r,\theta)$ ↑ smooth not rotational sym ex. $\mathbb{R}^2, S^2$



$\gamma(r,\theta)$ $r = r(t)$, $\theta = \theta(t)$

$$L = \int_0^{L_0} \sqrt{\left(\frac{dr}{dt}\right)^2 + g(r,\theta)^2 \left(\frac{d\theta}{dt}\right)^2}\, dt$$

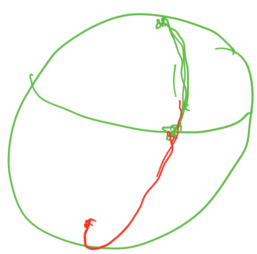
$$\geq \int_0^{L_0} \sqrt{\left(\frac{dr}{dt}\right)^2}\, dt$$

$$= L_0$$

segment

absolute

minimizes.

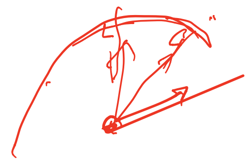


2nd order ODE

$\exists!$ sol $\underline{p}, \underline{\vec{v}}$

$u'' + [illegible]\, u = [illegible]$

$[illegible]'' + -$



Example

Polar Coordinates: $x = r\cos\theta$, $y = r\sin\theta$.

Then $dx = \cos\theta\ dr - r\sin\theta\ d\theta$, $dy = \sin\theta\ dr + r\cos\theta\ d\theta$

and $ds^2 = dx^2 + dy^2 =$

$(\cos\theta\ dr - r\sin\theta\ d\theta)^2 + (\sin\theta\ dr + r\cos\theta\ d\theta)^2$

which simplifies to

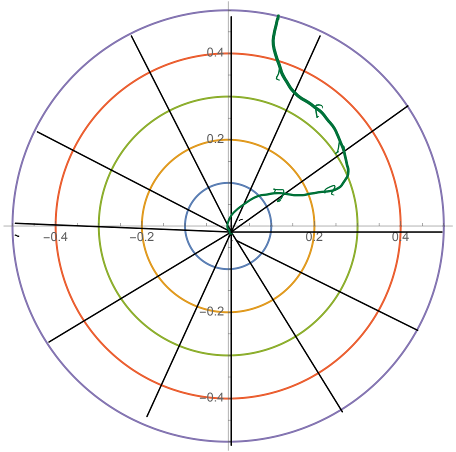
$$ds^2 = dr^2 + r^2\ d\theta^2, \tag{1}$$

Theorem

Let γ be a curve in $\mathbb{R}^2$ from the origin 0 to the circle C_R of radius R centered at 0.

Then

1. $L(\gamma) \geq R$.
2. *Equality holds if and only if gamma is a ray $\theta = const$*


0.4
0.2
−0.4
−0.2
0.2
0.4
−0.2
−0.4

Proof.

Let $\gamma(t) = (r(t), \theta(t), 0 \leq t \leq 1$. Then

$$L(\gamma) = \int_0^1 (r'(t)^2 + r(t)^2\theta'(t))^{\frac{1}{2}} dt \geq \int_0^1 r'(t) dt = R$$

and equality holds if and only if $\theta'(t) \equiv 0$. $\square$

Corollary

Given $p, q \in \mathbb{R}^2$, the shortest curve from p to q is the straight line segment $\overline{pq}$.

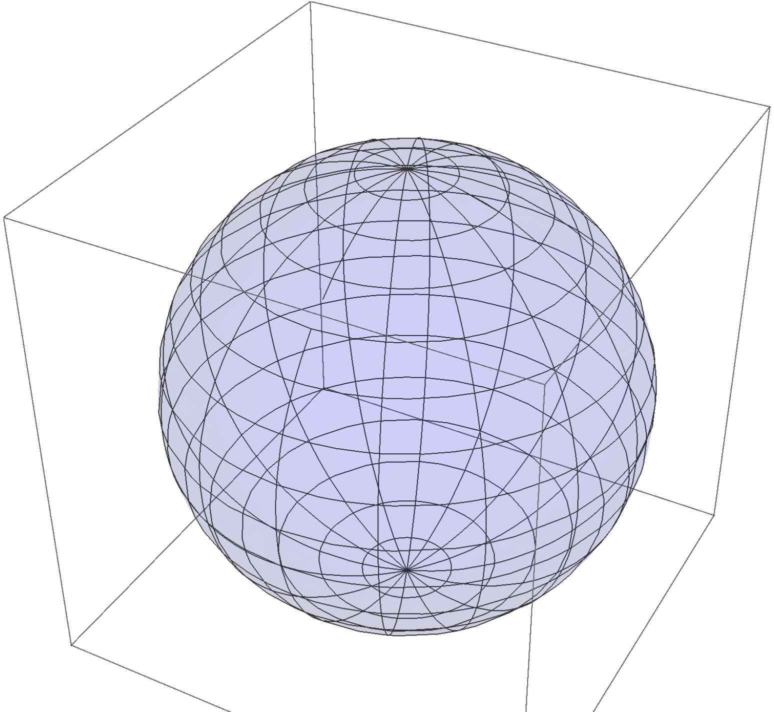
Example

Spherical coordinates:

$x = \sin\phi\cos\theta$, $y = \sin\phi\sin\theta$, and $z = \cos\phi$

$dx = \cos\phi\cos\theta\ d\phi - \sin\phi\sin\theta\ d\theta$,
$dy = \cos\phi\sin\theta\ d\phi + \sin\phi\cos\theta\ d\theta$, and
$dz = -\sin\phi\ d\phi$.

$$ds^2 = d\phi^2 + \sin^2\phi\ d\theta^2$$



Theorem

Let γ be a curve in S^2 from the north pole N to the "geodesic circle" $\phi = \phi_0$ of radius ϕ_0 centered at N, where $0 < \phi_0 < \pi$.

Then

1. $L(\gamma) \geq \phi_0$.
2. *Equality holds if and only if gamma is a great-circle arc $\theta = const$*

Proof.

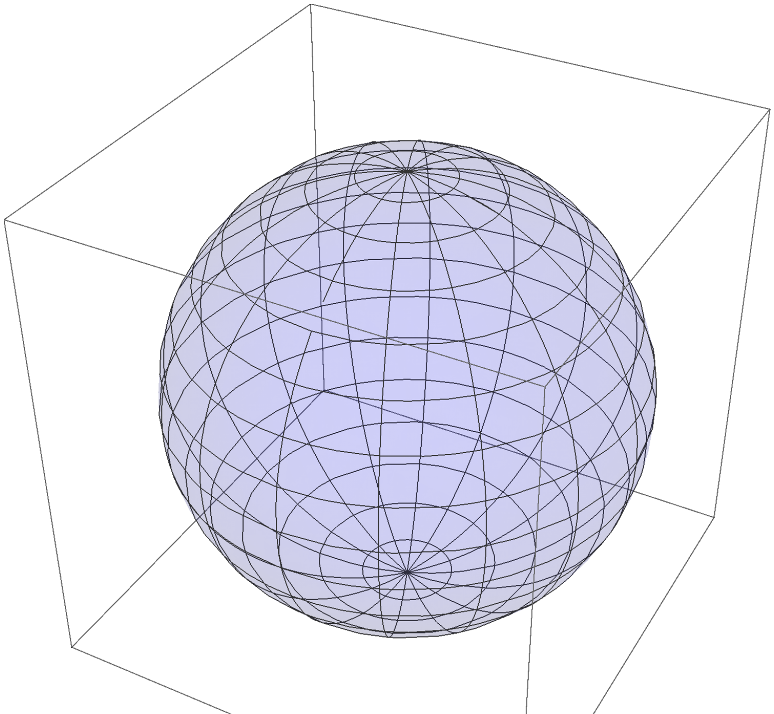
Let $\gamma(t) = (\phi(t), \theta(t)) 0 \leq t \leq 1$. Then

$$L(\gamma) = \int_0^1 (\phi'(t)^2 + \sin^2(\phi(t))\theta'(t))^{\frac{1}{2}} dt \geq \int_0^1 \phi'(t) dt = \phi_0$$

and equality holds if and only if $\theta'(t) \equiv 0$. $\square$

Corollary

1. *Given* $p, q \in S^2$*, not antipodal, the shortest curve from* p *to* q *is the shorter of the two great-circle arcs from* p *to* q
2. *If* p *and* q *are antipodal, there are infinitely many curves of shortest length from* p *to* q*.*



Geodesic Equation in Local Coordinates

This section was not covered in class this week, but results were used. more details next week.

- $\mathbf{x} : V \to U \subset S \subset \mathbb{R}^3$ as before.
- The vectors $\mathbf{x}_u, \mathbf{x}_v$ form a basis for $T_{\mathbf{x}(u,v)}S$
- $\gamma(s) = \mathbf{x}(u(s), v(s))$
- $\gamma'(s) = \mathbf{x}_u u' + \mathbf{x}_v v''$
- Differentiate once more:

$$\gamma'' = \mathbf{x}_u u'' + \mathbf{x}_v v'' + \mathbf{x}_{uu}(u')^2 + 2\mathbf{x}_{uv} u' v' + \mathbf{x}_{vv}(v')^2.$$

- To find γ''^T, note that the first two terms are tangential
- Write the sum of the last three terms as

$$a\mathbf{x}_u + b\mathbf{x}_v + \mathbf{n}$$

  with a, b scalar functions of u, v and $\mathbf{n}(u, v)$ normal.
- Then

$$\begin{pmatrix} (a\mathbf{x}_u + b\mathbf{x}_v + \mathbf{n}) \cdot \mathbf{x}_u \\ (a\mathbf{x}_u + b\mathbf{x}_v + \mathbf{n}) \cdot \mathbf{x}_v \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

- On the other hand, the first vector is also

$$\begin{pmatrix} (\mathbf{x}_{uu}(u')^2 + 2\mathbf{x}_{uv}u'v' + \mathbf{x}_{vv}(v')^2) \cdot \mathbf{x}_u \\ (\mathbf{x}_{uu}(u')^2 + 2\mathbf{x}_{uv}u'v' + \mathbf{x}_{vv}(v')^2) \cdot \mathbf{x}_v) \end{pmatrix}$$

- Therefore, letting $\mathbf{w} = \mathbf{x}_{uu}(u')^2 + 2\mathbf{x}_{uv}u'v' + \mathbf{x}_{vv}(v')^2$,

$$\begin{pmatrix} \mathbf{w} \cdot \mathbf{x}_u \\ \mathbf{w} \cdot \mathbf{x}_v \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

- Therefore, if

$$\begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1}$$

- We get

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} \begin{pmatrix} \mathbf{w} \cdot \mathbf{x}_u \\ \mathbf{w} \cdot \mathbf{x}_v \end{pmatrix}$$

- Writing

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \Gamma^1_{11} u'^2 + 2\Gamma^1_{12} u' v' + \Gamma^1_{22} v'^2 \\ \Gamma^2_{11} u'^2 + 2\Gamma^2_{12} u' v' + \Gamma^2_{22} v'^2 \end{pmatrix}$$

- We get that the components of γ''^T in the basis $\mathbf{x}_u, \mathbf{x}_v$ are

$$\begin{pmatrix} u'' + \Gamma^1_{11} u'^2 + 2\Gamma^1_{12} u' v' + \Gamma^1_{22} v'^2 \\ v'' + \Gamma^2_{11} u'^2 + 2\Gamma^2_{12} u' v' + \Gamma^2_{22} v'^2 \end{pmatrix}$$

- In particular the geodesic equation is a system of second order ODE's

$$
\begin{aligned}
u'' + \Gamma^1_{11} u'^2 + 2\Gamma^1_{12} u' v' + \Gamma^1_{22} v'^2 &= 0 \\
v'' + \Gamma^2_{11} u'^2 + 2\Gamma^2_{12} u' v' + \Gamma^2_{22} v'^2 &= 0
\end{aligned}
$$

  where the six coefficients $\Gamma^i_{jk} = \Gamma^i_{jk}(u, v)$ are smooth functions on U, and the coefficients of u'', v'' are $\equiv 1$.
- Write $\mathbf{u} = \mathbf{u}(s) = ((u(s), v(s))$ for a solution of the system
- Write p for a point in U and $\mathbf{v}$ for a vector in $\mathbb{R}^2$, which we think of as a tangent vector to U at p.

Standard existence and uniqueness theorem for such a system of second oreder ODE's:

Theorem

- *Given any* $p_0 \in U$ *and any* $\mathbf{v}_0 \in \mathbb{R}^2$ *there exist*
    1. *A nbd* W *of* $(p_0, \mathbf{v}_0)$ *in* $U \times \mathbb{R}^2$
    2. *An interval* $(-a, a) \subset \mathbb{R}$
- *So that for any* $(p, \mathbf{v}) \in W$ *there exists a unique solution* $\mathbf{u}(s)$ *of the system satisfying the initial conditions* $\mathbf{u}(0) = p$ *and* $\mathbf{u}'(0) = \mathbf{v}$. *Call this solution* $\mathbf{u}(s, p, \mathbf{v})$,
- *It depends smoothly on the initial conditions* $p, \mathbf{v}$ *in the sense that the map* $\mathbf{u} : (-a, a) \times W \to U$ *given by* $(s, p, \mathbf{v}) \mapsto \mathbf{u}(s, p, \mathbf{v})$ *is smooth.*

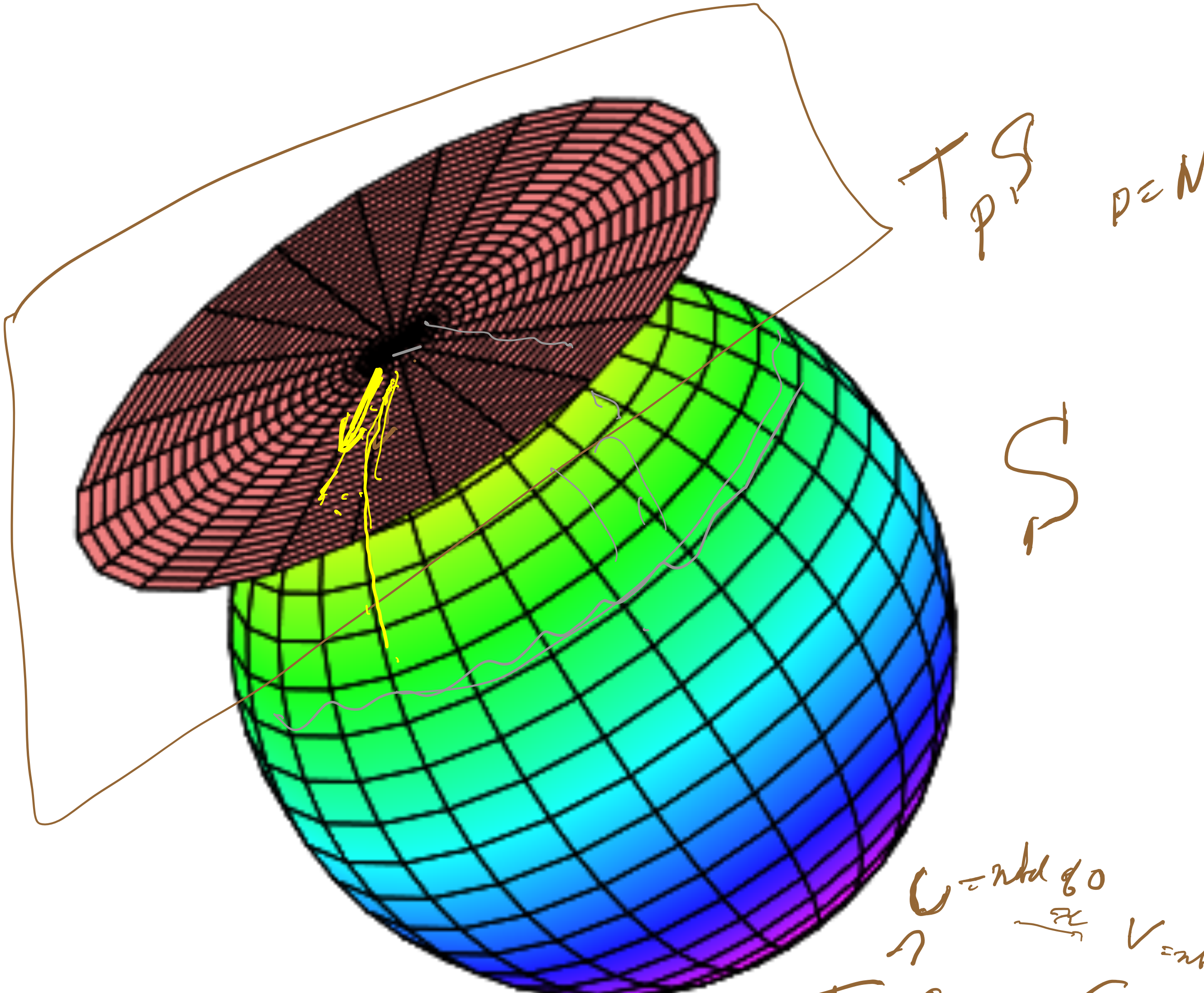
Some Properties of the Solutions

- May assume $p = 0$ and $d_0\mathbf{x}$ is an isomtry (equivalently, $(g_{ij}(0)) = I$ unit matrix)
- Uniqueness of solutions gives

$$\mathbf{u}(rs, p, \mathbf{v}) = \mathbf{u}(s, p, r\mathbf{v}) \ \text{ for any } r \in \mathbb{R}$$

- Enough to consider solutions with $|\mathbf{v}(0)| = 1$, at the expense of changing interval of existence.
- or any $\mathbf{v}_0$ so that $|\mathbf{v}_0| = 1$, there exists a neighborhood V of $\mathbf{v}_0$ and an $a > 0$ so that the solution $\mathbf{u}(s, 0.\mathbf{v})$ exists for all $(s, \mathbf{v}) \in (-a, a) \times V$
- Use compactness of S^1 to cover by finitely many V and take $b =$ minimum a.

- Lemma
  There exists $b \in (0, \infty]$ *so that the solution* $\mathbf{u}(s, 0, \mathbf{v})$ *of the geodesic equation is defined for all* $(s, \mathbf{v}) \in (-b, b) \times S^1$*.*
  - In other words, for any fixed length $c < b$ all geodesics through 0 in all directions $\mathbf{v} \in S^1$ are defined up to c.
  - $b = \infty$ is possible, in fact, it is the ideal situation.



U = nbhd of 0 $\xrightarrow{\cong}$ V = nbhd of p

$\cap$

$T_pS \longrightarrow S$

$v \in U \mapsto \gamma(1)$

where $\boxed{\gamma(t)}$ = geod starting at

p, $\gamma'(0) = v$ (- travels at speed $|v|$)

~~h = geod~~

time $|v|$ indeed $h'(0) = \dfrac{v}{|v|}$

Fact Can do this for any surface

Follows ODE applied to geod equation

First explain consequences

$$\begin{tikzcd} U \arrow[r, "x"] \arrow[d, phantom, "\cup" description] & V \arrow[d, phantom, "\cap" description] \\ T_pS & S \end{tikzcd}$$

called exponential map

$\exp_p : U \to V$.

"Normal coordinates"

$\exp : T_pS \to S$

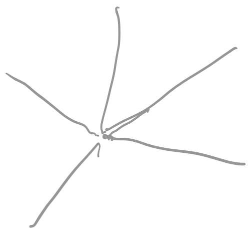
rectangular coord u, v to T_pS

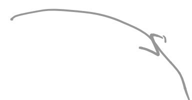
or polar coord r, θ

$r, \theta \to \exp(r\cos\theta, r\sin\theta)$

(r, θ)

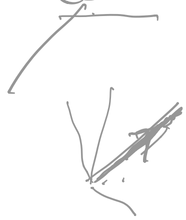
" geodesic polar coordinates





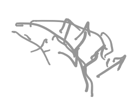
$$(d\exp \cdot d\exp) = g_{11}(u,v)\,du^2 + 2g_{12}(u,v)\,du\,dv + g_{22}(u,v)\,dv^2$$

$$(E)\,dr^2 + (F)\,dr\,d\theta + (G)\,d\theta^2$$



$E = 1$

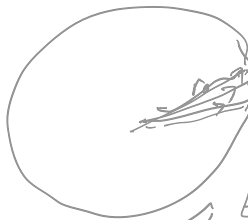
F ? G ?



$$\frac{\partial}{\partial\theta} \exp(r, \theta) \Big|_{\theta = \theta_0}$$

$= V(r) =$ variation fld.

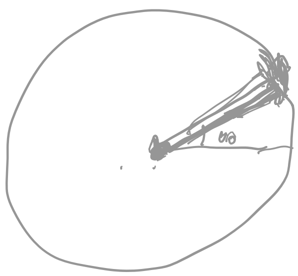
First Variation:



$$\left.\frac{d L(\exp(r,\theta))}{d\theta}\right|_{\theta=\theta_0} = -\underbrace{\int_0^{r_0} V\cdot\frac{D\gamma'}{Ds}\,ds}_{=0} + \left. V\cdot\gamma'\right|_0^{r_0}$$

$$= V(r_0)\cdot\gamma'(r_0) - \underbrace{V(0)\cdot\gamma'(0)}_{=0}$$

$\gamma = \exp(s, \theta_0)$



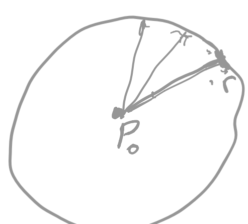
$\Rightarrow V\cdot\gamma'(r_0) = 0$

~~geod circle of radius r at p_0~~

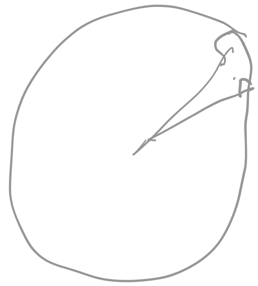
~~○ → pts at dist r from p_0~~

~~Gauss Lemma:~~

curve $\exp(r_0, \theta)$ for fixed r_0:



~~Gauss Lemma~~ Geodesic circles centered at p are ⊥ geodesics thru P.



$$dr^2 + 2F\,dr\,d\theta + G\,d\theta^2$$

$$\frac{\partial\,\text{sub}}{\partial r}\cdot\frac{\partial\,\text{sub}}{\partial \theta} = 0$$

$$dr^2 + G\,d\theta^2 \qquad G \geq 0$$

$$G = g(r,\theta)^2$$

$$g(r,\theta) \geq 0$$

$$E\,\cancel{(r,\theta)} = \left|\frac{\partial\,\text{sub}}{\partial r}\right|^2$$

Any surf S, any pt $p \in S$

suff small nbd have geod polar coords

$$dr^2 + g(r,\theta)^2\,d\theta^2$$

"metric"

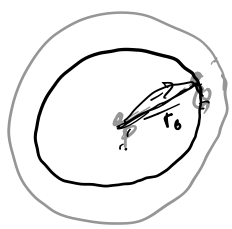
Riemannian metric!

$$\frac{\partial\,\text{sub}}{\partial r} \quad \frac{\partial\,\text{sub}}{\partial \theta}$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \frac{\partial\,\text{sub}}{\partial r}\cdot\frac{\partial\,\text{sub}}{\partial r} & \frac{\partial\,\text{sub}}{\partial r}\cdot\frac{\partial\,\text{sub}}{\partial \theta} \\ \cdot & \frac{\partial\,\text{sub}}{\partial \theta}\cdot\frac{\partial\,\text{sub}}{\partial \theta} \end{pmatrix}$$

Consequence: geodesic segments from 0 in geod. polar coords are absolute minimizing curves

Means: p, U



Any curve γ in U from P to Q

$r(t), \theta(t)$

$$L = \int_0^{r_0} \|\gamma'(t)\| \, dt$$

$$= \int_0^{r_0} \sqrt{\left(\frac{dr}{dt}\right)^2 + g(r,\theta)^2 \left(\frac{d\theta}{dt}\right)^2} \, dt$$

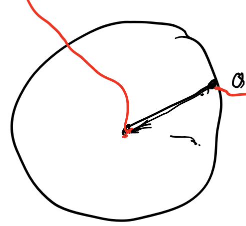
$$\geq \int_0^{r_0} \sqrt{\left(\frac{dr}{dt}\right)^2} \, dt$$

$$\boxed{= \int_0^{r_0} \left|\frac{dr}{dt}\right| dt = r_0 \qquad \geq \int_0^{r_0} \frac{dr}{dt} \, dt = r_0}$$

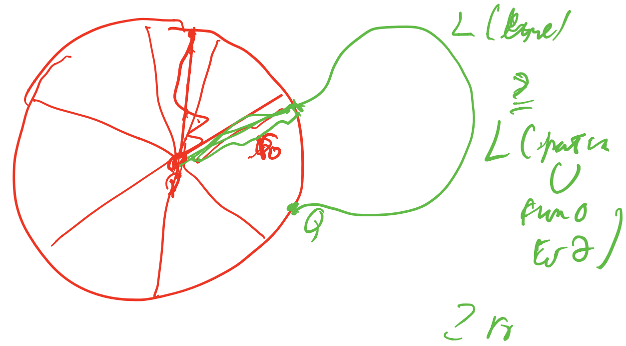
$L(\gamma) \geq r_0$, $= \Leftrightarrow \frac{d\theta}{dt} = 0$

$=$ geod

$\theta = $ const



Proved length of any curve in U
from P to ∂ circle $\geq r_0$



$$dr^2 + g(r,\theta)^2 d\theta^2$$

What's g?

C_r = geodesic circle of radius r centered at P.

$$\boxed{L(C_r) = \int_0^{2\pi} g(r,\theta)\,d\theta}$$

Fact

$$g(r,\theta) = c_0(\theta) + c_1(\theta) r + c_2(\theta) r^2 + c_3(\theta) r^3 + \underbrace{O(r^4)}_{|\ | \leq C r^4}$$

Fact: $c_0 = 0$

$c_1 = 1$

$c_2 = 0$

$c_3(\theta)$ = indep of $\theta = c_3$

$$L(C_r) = \underbrace{2\pi r}_{\text{Euclidean}} + 2\pi(c)\, r^3 - -$$

Ex: $dr^2 + \sin^2 r\, d\theta^2$, $\quad \underline{r = \varphi}$

$$\sin r = r - \frac{K r^3}{3!} \cdots$$

$$c = \frac{-K}{6}$$

__Def__ $K(p) = -6C$

Called the Gaussian curvature of S^2 at p.

$$c = \frac{-1}{6} \quad K = -6\left(\frac{-1}{6}\right) = 1$$

Controls leading term in deviation from Euclidean.