Introduction to Algebraic and Geometric Topology
Week 10

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Fall 2017

Recall: Quotient Topology

$X \to Y$

- **Definition**

  Let

  1. $(X, \mathcal{T}_X)$ be a topological space,
  2. Y a set
  3. $f : X \to Y$ a *surjective* map.

  The *quotient topology* $\mathcal{T}_Y$ on Y, is defined as

$$T_Y = \{U \subset Y \mid f^{-1}(U) \in \mathcal{T}_X\}$$

- T_Y is the *largest* topology on Y that makes f continuous.

f surjective

Recall: Identification

- **Definition**

  Let

  1. $f : (X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces.
  2. $f : X \to Y$ a surjective map.

  f is called an *identification* if and only if T_Y is the quotient topology just defined:

$$T_Y = \{U \subset Y \mid f^{-1}(U) \in \mathcal{T}_X\}$$

- Equivalent statements: A surjective map $f : X \to Y$ is an identification if and only if
  - $$U \text{ open in } Y \overset{\text{cont}}{\Longleftrightarrow} f^{-1}(U) \text{ is open in } X$$
    $\Leftarrow$ by our top
  - $$F \text{ closed in } Y \Longleftrightarrow f^{-1}(F) \text{ is closed in } X$$

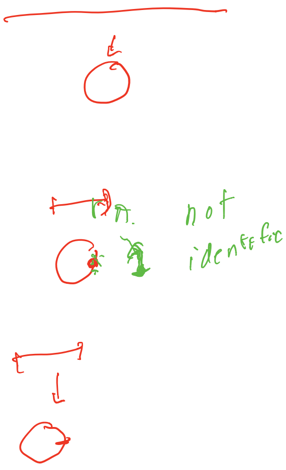
Recall: Examples

1. $f_1 : \mathbb{R} \to S^1$ defined by $f(t) = (\cos t, \sin t)$.

2. $f_2 : [0, 2\pi) \to S^1$ same formula.

3. $f_3 : [0, 2\pi] \to S^1$ same formula.

1 and 3 are identifications, 2 is not.



Sufficient Conditions for Identification

1. Recall definition ot *open map* and *closed map*

   U open $\Rightarrow f(U)$ open

   F closed $\Rightarrow f(F)$ closed.

2. $f : X \to Y$ continuous, surjective and *open* $\Longrightarrow$ identification. $\Rightarrow$ id

   $$f[\underbrace{f^{-1}(U)}_{\text{open}}] = \underbrace{U}_{\text{open}}$$

   open map

3. $f : X \to Y$ continuous, surjective and *closed* $\Longrightarrow$ identification.

$$\boxed{f(f^{-1}(A)) \subset A}$$

$= ?$

$f(x) \in A$

suff:

f surjection

$\Rightarrow f(f^{-1}(A)) \supset A$

$y \in A$

$\exists x \quad f(x) = y$

$x \in f^{-1}(A)$

$A \subset f(f^{-1}(A))$

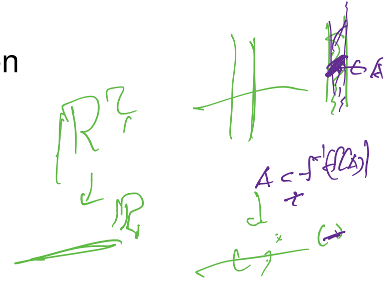
Checking Identifications

- Useful facts:
- Suppose $f : X \to Y$, $A \subset X$, $B \subset Y$. Then

1. $f(f^{-1}(B)) \subset B$
2. If f is surjective, $f(f^{-1}(B)) = B$
3. $A \subset f^{-1}(f(A))$.
4. If f is surjective, then

$$\boxed{A = f^{-1}(B)} \text{ for some } B \subset Y \iff \boxed{A = f^{-1}(f(A))}$$

and, in this case, $B = f(A)$.



Continuous Maps

always

- X, Y, Z topological spaces.
- $f : X \to Y$ identification,
- $g : Y \to Z$ a map.
- Then g is continuous $\iff$ $g \circ f$ is continuous

needs identifi

$$
\begin{tikzcd}
X \arrow[r, "g \circ f"] \arrow[d, "f"'] & Z \\
Y \arrow[ur, "g"'] &
\end{tikzcd}
$$

- Proof:
- If $U \subset Z$, then

$$(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$$

- Thus, if $U \subset Z$,

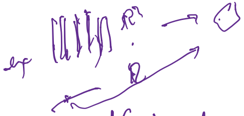
$$g^{-1}(U) \text{ is open } \iff (g \circ f)^{-1}(U) \text{ is open}$$

g cont

g∘f conti

- Thus g continuous $\iff g \circ f$ is continuous.

- Equivalent Formulation:
- X, Y, Z topological spaces, $f : X \to Y$ an identification.
- $h : X \to Z$ a map that is constant on the fibers $f^{-1}(y)$ of f.
- Then the map g in the following diagram is defined:



$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"'] & Z \\ Y \arrow[ur, "g"'] & \end{tikzcd}$$

$h(x)$ depends only on $f(x)$

$h(x,y)$ indep of y

$f(x) = h(x,y)$ makes sense

- g is continuous $\iff$ h is continuous.

period 2π

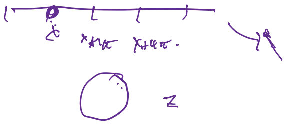
$f(x+2\pi) = f(x)$

$\Downarrow$

$f(x+2\pi n) = f(x)$

$\forall n \in \mathbb{Z}$

- Example: Periodic functions $h : \mathbb{R} \to \mathbb{R}$



$$\begin{tikzcd} \mathbb{R} \arrow[r, "h"] \arrow[d, "f"'] & \mathbb{R} \\ S^1 \arrow[ur, "g"'] & \end{tikzcd}$$

$z = e^{it}$

$g(z) = f(t)$

$e^{it} = z$

where $f(t) = (\cos t, \sin t)$.

Cont periodic funcs on $\mathbb{R}$

$\longleftrightarrow$ Cont funcs S^1

- Example: Doubly periodic functions $h : \mathbb{R}^2 \to \mathbb{R}$

  $h(s+2\pi, t) = h(s, t)$ and $h(s, t+2\pi) = h(s, t)\ \forall (s, t) \in \mathbb{R}^2$
- Let T be the *torus* $S^1 \times S^1$

  $$T = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1^2 + x_2^2 = x_3^2 + x_4^2 = 1\}$$

  and $f : \mathbb{R}^2 \to T$ be defined by

  $$f(s, t) = (\cos s, \sin s, \cos t, \sin t).$$
- Then f is an identification and h is continuous $\iff$ g is continuous:

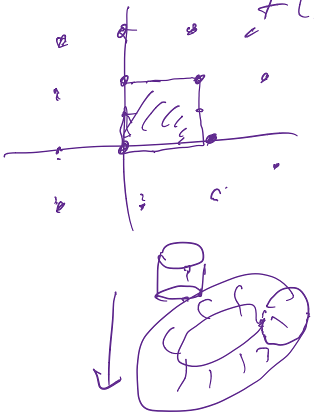
$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "h"] \arrow[d, "f"'] & \mathbb{R} \\ T \arrow[ur, "g"'] & \end{tikzcd}$$

$\mathbb{R}^2 \longrightarrow \mathbb{R}$

$f(s+2\pi, t) = f(s,t)$

$f(s, t+2\pi) = f(s,t)$

$$f(s+2\pi m,\ t+2\pi n) = f(s,t)$$



Torus

$$(s,t) \sim (s',t')$$

$$\iff s'-s \in \mathbb{Z}$$

$$t'-t \in \mathbb{Z}$$

$$(s',t')-(s,t) \in \mathbb{Z}\times\mathbb{Z}$$

$$\mathbb{R}^2/\sim \ = \mathbb{R}^2/\mathbb{Z}^2$$

Cont. doubly periodic func on $\mathbb{R}^2$

$\iff$ Cont functions $S^1 \times S^1$ = Torus.

Equivalence Relations

- $f : X \to Y$ surjective map of sets $\iff$

  equivalence relation on X:

$$x_1 \sim x_2 \iff f(x_1) = f(x_2).$$

$x \sim x$

$x \sim y \iff y \sim x$

$x \sim y$ & $y \sim z \Rightarrow x \sim z$

$f : X \to$ set of equiv classes

- $f : X \to Y$ surjective map of sets $\iff$

  Partition of X into disjoint subsets

$$X = \coprod_{y \in Y} f^{-1}(y)$$

$Y = X/\sim$

set of equiv classes

Connected Components

- Let X be a topological space. Define a relation

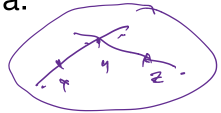
$$\boxed{x \sim y \iff \exists \text{ a connected } C \subset X \text{ with } x, y \in C}$$

- **Theorem**
  The relation just defined is an equivalence relation.

- **Proof.**
  Clearly $x \sim x$ and $x \sim y \iff y \sim x$.
  Transitivity $x \sim y$ and $y \sim z \implies x \sim z$ follows from the next lemma. □

$x \in \{x\}$ conn.

$x, y \in C \iff y, x \in C.$

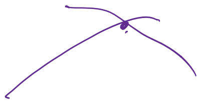


$x \sim z$

Lemma

trans.

If $C_1, C_2 \subset X$ are connected and $C_1 \cap C_2 \neq \emptyset$, then $C_1 \cup C_2$ is connected.



let $\varphi : C_1 \cup C_2 \to \{0,1\}$

cont func.

$\varphi|_{C_1} = \text{const} = c_1$

$\varphi|_{C_2} = \text{const } c_2$

$x \in C_1 \cap C_2 \Rightarrow \varphi(x) = c_1$
$= c_2$

$\Rightarrow c_1 = c_2 = \varphi(x)$

Better Lemma:

Lemma

- *Let $\{C_\alpha\}_{\alpha\in A}$ be a collection of connected subsets of X indexed by a set A. Suppose that $\cap_\alpha C_\alpha \neq \emptyset$. Then $\cup_\alpha C_\alpha$ is connected.*
- *Suppose $C \subset X$ is connected. Then its closure $\overline{C}$ is connected.*

Same pf: $\varphi : \bigcup C_\alpha \to \{0,1\}$

$\varphi|_{C_\alpha}$ is cnst $= c_\alpha$

if $x \in \bigcap C_\alpha$

$c_\alpha = x$ $\forall \alpha$

φ cnst

$\varphi : \overline{C} \to \{0,1\}$

$\varphi|_C = c \Rightarrow \varphi|_{\overline{C}} = c$

- ## Definition

  The equivalence classes of the equivalence relation just defined are called the *connected components* of X.

  - If $x \in X$, let C_x be the connected component of X containing x.

  ## Theorem

  - *C_x is the largest connected subset of X containing x.*
  - *C_x is closed in X.*

C conn, $x \in C$, then $\cancel{x \in C_x}$ $C \subset C_x$

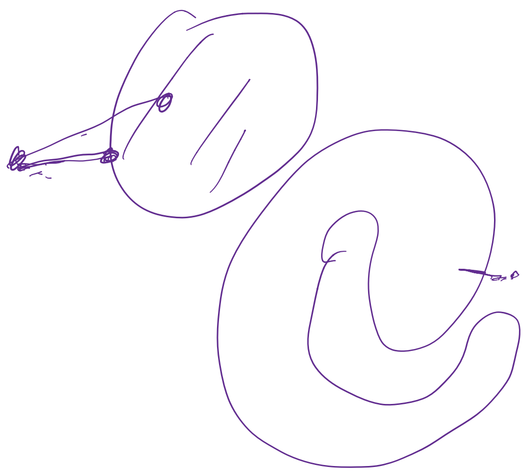
$C_x = \bigcup \{C : C \text{ conn}, x \in C\}$

$\cancel{x} \in \bigcap \{C : C \text{ conn}, x \in C\}$

$d(x, A) = \inf \{ d(x,y) : y \in A \}$

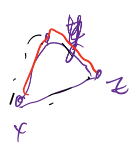
$\shortparallel$

$f(x)$



Prob f is cont | easier: f Lipschitz

$\triangle$' ineq $\quad d(x,z) \le d(x,y) + d(y,z)$



$\longrightarrow \quad d(x,z) - d(y,z) \le d(x,y)$

Interchange x & y

$\longrightarrow \quad d(y,z) - d(x,z) \le d(x,y)$

$\Longleftrightarrow \quad \boxed{|d(x,z) - d(y,z)| \le d(x,y)}$

$g_z(x) = d(x, z)$

$x \to z$

$$\boxed{|g(x) - g(y)| \le d(x,y)}$$

$d(x, z)$

$\forall z \in X$, have fcn $g_z : X \to \mathbb{R}$

$g_z(x) = d(x, z)$

$d(x, A) = \inf \{ g_z(x) : z \in A \}$

(Thm: $\mathcal{F}$ a family of cont fns

$\underline{f}(x) = \inf \{ f(x) : f \in \mathcal{F} \}$

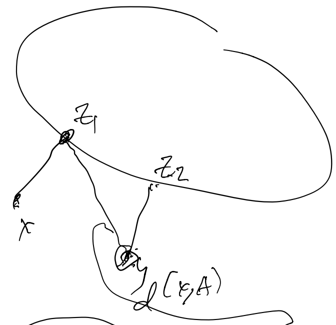
cont)

$d(x, A)$

$\underline{d(x, A)} - \underline{d(y, A)}$

Suppose $d(x, A) = d(x, z_1)$

$d(y, A) = d(y, z_2)$



$d(y,z_1) = d(y,A)$

$d(x,z_1) - d(y,z_1) \le d(x,y)$ ④

$d(y,z_1) - d(x,z_1) \le d(x,y)$

② $d(x,A) - d(y,A) \le d(x,y)$

$d(y,A) - d(x,A) \le d(x,y)$

$d(x,z_2) - d(y,z_2) \le d(x,y)$

$d(y,z_2) - d(x,z_2) \le d(x,y)$

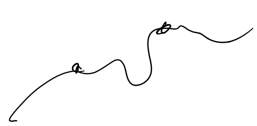
$d(y,A) - d(x,A) \le d(y,z_1) - d(x,z_1) \le d(x,y)$

$\le d(x,y)$

Connected Components of $(X, \mathcal{T}_X)$

$x \sim y \iff \exists$ Connect $C \subset X$

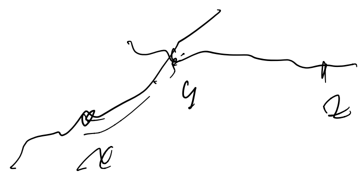
s.t. $x, y \in C$



$x \sim x$

$x \sim y \Leftrightarrow y \sim x$ } clear

$x \sim y \ \& \ y \sim z \Rightarrow x \sim z$



$x, y \in C_1$

$y, z \in C_2$

$\exists$ conn set C_3

$x, z \in C_3$

$C_3 = C_1 \cup C_2$

C_1, C_2 conn & $C_1 \cap C_2 \neq \emptyset$

$\Rightarrow C_1 \cup C_2$ conn

$\varphi : C_1 \cup C_2 \to \{0,1\}$ cont

$\varphi|_{C_1} = \text{const } c_1$

$\varphi|_{C_2} = \text{const } c_2$

$\varphi|_{C_1}(y) = \varphi|_{C_2}(y) \Rightarrow c_1 = c_2$

const.

More

$\{C_\alpha\}_{\alpha \in A}$ any collection of conn subsets of X

if $\bigcap_{\alpha \in A} C_\alpha \neq \phi$

$\Rightarrow \bigcup_{\alpha \in A} C_\alpha$ conn.

Same pf: $\varphi : \bigcup C_\alpha \to \{0,1\}$ cont

let $\varphi_\alpha = \varphi|_{C_\alpha}$

$\varphi_\alpha = \text{const } c_\alpha$

$x_0 \in \bigcap C_\alpha$

$\underbrace{\varphi_\alpha(x_0)}_{\text{same } \forall \alpha} = c_\alpha$

Conn Comp

Definition

Let $\boxed{\pi_0(X)}$ denote the set of connected components of X.

- There is a surjective map $\pi : X \to \pi_0(X)$ defined by

$$\pi(x) = C_x,$$

  the connected component of X containing x.
- $card(\pi_0(X)) = 1 \iff X$ is connected.
- In general, $card(\pi_0(X))$ is the number of connected components of X.
- $\pi_0(X)$ is a topological space, with the quotient topology.

Ex: If X loc conn $\Rightarrow \pi_0(X)$ is discrete

$$\begin{array}{ccc} X & \longrightarrow & Y \qquad C_{f(x)} \\ \cup & & \quad \cup \\ C_x & \longrightarrow & f(C_x) \ni f(x) \end{array}$$

- If $f : X \to Y$ is continuous, there is a map, first of sets,

$$f_* : \pi_0(X) \to \pi_0(Y)$$

"functor"

defined by

$$f_*(C_x^X) \to C_{f(x)}^Y$$

where C_x^X and C_y^Y denote the connected components of x in X and of y in Y respectively.

- *Exercise 1*: f_* is well-defined:
$x \sim x' \implies C_{f(x)}^Y = C_{f(x')}^Y$.

- Reason: $x \sim x' \iff C_x^X = C_{x'}^X$
  and $f(C_x^X)$ is a connected subset of Y contaning $f(x)$, hence
  $f(C_x^X) \subset C_{f(x)}^Y$.
- *Exercise 2*: $\boxed{f \text{ homeomorphism} \implies f_* \text{ bijective.}}$
- *Exercise 3*: f_* is continuous.
- *Exercise 4*: f homeomorphism $\implies f_*$ homeomorphism.

Examples

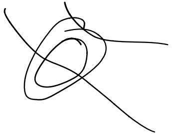
- $\pi_0(\mathbb{R}^n)$ has cardinality one for $n = 1, 2, \ldots$.
- If X is discrete $\pi : X \to \pi_0(X)$ is a homeomorphism.
- *Homework Problem* : X locally connected $\implies \pi_0(X)$ is discrete.

Theorem

For $n \geq 2$, $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^n$. $n \geq 2$

$\mathbb{R}$ not homeo to $\mathbb{R}^2$

f_x



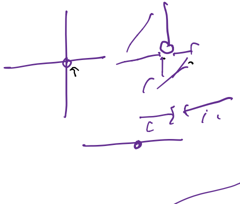
Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ homeo

$0 \in \mathbb{R}^2 \to f(0)$ in $\mathbb{R}$

$\mathbb{R}^2 - 0 \xrightarrow{f|_{\mathbb{R}^2 - 0}} \mathbb{R} - f(0)$ is a homeo

$\mathbb{R}^2 - (0)$ conn : path conn

$\mathbb{R} - f(0)$ not conn



- Same argument: the unit interval I is not homeomorphic to $I \times I$.
- This was used in the proof that the Euclidean and taxi-cab metrics not being isometric.

$\pi_0(\mathbb{R}^2 - \{0\})$ card 1

$\pi_0(\mathbb{R} - f(0))$ card 2

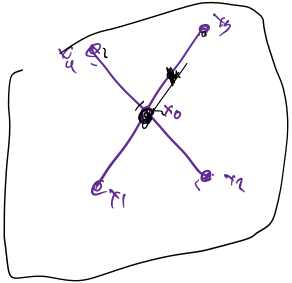
not homeo

(promised earlier)

Theorem

Let X be the subspace of $\mathbb{R}^2$ of the letter "X". Let x_0 be the junction point of the four "branches" of X, and let x_1, x_2, x_3, x_4 be the other endpoints of the branches. Let $f : X \to X$ be a homeomorphism.

Then $f(x_0) = x_0$ and f permutes the points $x_1, \ldots, x_4$.



Homeo $(X) \circlearrowleft$

$f(x_0) = x_0$

x_0 only pt w/ $\# \pi_0(X - \{x_0\}) = 4$

$x_1 - - x_4$ only pt $\# \pi_0(X - \{x_i\}) = 1$

x_0 . unique pt in X s.t. $\# \pi_0(X - x_0) = 4$

- If X is the Cantor set, what is $\pi_0(X)$?
- Some information in the homework.

?

$$\# \pi_0(X - x) = \begin{cases} 4 & \text{if } x = x_0 \\ 1 & \text{if } x = x_1, x_2, x_3, x_4 \\ 2 & \text{otherwise.} \end{cases}$$

if $f : X \to X$ homeo $\Rightarrow$ ~~$f(x)$~~

$\# \pi_0(X - f(x))$ has to be same #!

$= \# \pi_0(X - x)$

- Let's go back to the definition of f_*:
- If $f : X \to Y$ is continuous, there is a map, first of sets,

$$\boxed{f_* : \pi_0(X) \to \pi_0(Y)}$$

defined by

$$f_*(C_x^X) \to C_{f(x)}^Y$$

where C_x^X and C_y^Y denote the connected components of x in X and of y in Y respectively.

- *Exercise 1*: f_* is well-defined: $x \sim x' \implies C_{f(x)}^Y = C_{f(x')}^Y$.

▸ Better to first say:

Theorem

1. *The connected components of X are the maximal connected subsets of X.*
2. *Every non-empty connected subset of X is contained in a unique maximal connected subset.*

C_0 = conn set $\subset X$
$\neq \phi$

$C_2 = \bigcup \{ C \subset X ; \, C_0 \in C \}$ conn'd

Connected, Max

- Then $C^Y_{f(x)}$ is *the* maximal connected subspace of Y containing $f(C_x)$.
- So is $C^Y_{f(x')}$ for any $x' \in C_x$.
- So $C^Y_{f(x)} = C^Y_{f(x')}$

$\{0,1\}^{\mathbb{N}}$ Cantor set? Totally disconnected

$C_x = \{x\}$ $\forall x \in X$.

- In the same spirit as looking at homeomophisms of the letter X:
- *Exercise*: Classify the (capital) letters $A, B, C, \ldots Z$ of the Roman alphabet up to homeomorphism.

A B C D E

HW

X compact Hausdorff

$A, B \subset X$ closed subsets

$A \cap B = \phi$

$\Rightarrow \exists$ opensets U, V

$\cancel{A \subset X}$ $A \subset U$, $B \subset V$

$U \cap V = \phi$

Quotient Topology

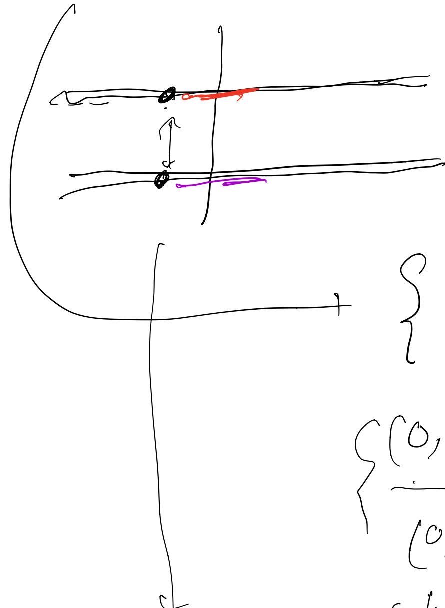
Connected Components

Quotients of Hausdorff

Spaces need not be Hausdorff

Example

$X = \mathbb{R} \sqcup \mathbb{R} =$

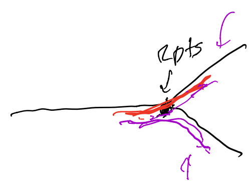


$\{(0,x) : x \in \mathbb{R}\} \cup \{(1,x),\ x \in \mathbb{R}\}$

$(0,x) \sim (1,x)$ if $x < 0$

$(0,x)$ $x \geq 0$

$(1,x)$



$(0,0)$, $(1,0)$

$[(0,x)] = \begin{cases} \{(0,x),(1,x)\} & \text{if } x<0 \\ \{(0,x)\} & \text{if } x \geq 0 \end{cases}$

$[(1,x)] = \begin{cases} \{(0,x),(1,x)\} & \text{if } x<0 \\ \{(1,x)\} & \text{if } x \geq 0 \end{cases}$

$[(0,0)]$ & $[(1,0)]$ don't have disjoint nbhds

~~More~~ Another Ex:

$$\mathbb{R} \xrightarrow{\phi} \mathbb{R}/\mathbb{Q}$$

$x \sim y \iff x - y \in \mathbb{Q}$

$U \subset \mathbb{R}/\mathbb{Q}$

$\mathbb{R}$

$\phi^{-1}(U)$

$U = \mathbb{R}$

if $x \in \phi^{-1}(U)$

$x + r \in \phi^{-1}(U)$ $\forall r \in \mathbb{Q}$

Surfaces

Definition

A *topological surface* is a Hausdorff space with a countable basis that is locally homeomorphic to $\mathbb{R}^2$.

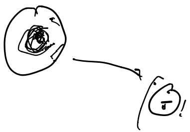
(connected)

Surface $\longleftrightarrow$ locally homeo to $\mathbb{R}^2$

X locally homeo to $\mathbb{R}^2$

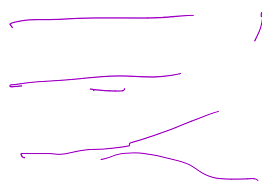
$\forall x \in X$ $\exists$ nbr U of x

s.t. U ~~homeo~~ $V \subset \mathbb{R}^2$



Hausdorff

countable basis $\leftarrow$ no more

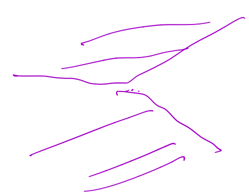


$\mathbb{R} \sqcup \mathbb{R}$

$\{ (1,x) \sim (0,x) \quad x<0 \}$

product with $\mathbb{R}$

$(0,x,z)$
$\sim (1,x,z)$ if $x<0$



$[(0,0,z)]$
$[(1,0,z)]$ no disjoint nbhd

non-Hausdorff space
loc homeo to $\mathbb{R}^2$

Examples

- An open subset of $\mathbb{R}^2$
- The unit sphere $S^2 \subset \mathbb{R}^3$

$$\cancel{X}$$

$$x \in \overline{U} \xrightarrow{\varphi} \overline{V} \subset \mathbb{R}^2$$

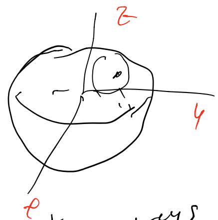
h

any N nbhd of x

$$x \in N \subset U$$

$$\varphi|_N : N \to \varphi(N) \subset \mathbb{R}^2$$

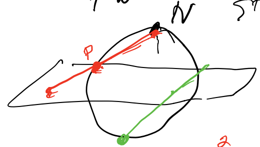
homeo



S^2 locally homeo to $\mathbb{R}^2$

Two ways

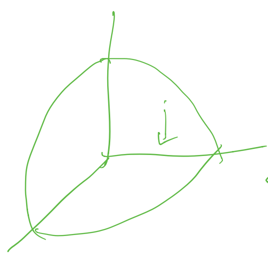
N ster proj from N



$P \to \{NP \cap \mathbb{R}^2 : z = 0\}$

$S^2 - N \xrightarrow{\text{homeo}} \mathbb{R}^2$

$S^2 - S \xrightarrow{\text{hm}} \mathbb{R}^2$



pr.

$\{z > 0\} \to$

$\left.\begin{array}{l} x^2+y^2+z^2=1 \\ z>0 \end{array}\right\} \to (x,y) \in \mathbb{R}^2 \ (z=0)$

$\searrow \{(x,y) \in \mathbb{R}^2 : x^2+y^2<1\}$

$x, y, \sqrt{1-x^2-y^2} \leftarrow (x,y)$

Same with $z<0$

6 open sets $\left\{\begin{array}{l} z>0 \\ z<0 \\ x>0 \\ x<0 \\ y>0 \\ y<0 \end{array}\right.$

6 sets each homeo to $\{x^2+y^2<1\}$

cover S^2

$f(x,y,z) = 1-x^2-y^2-z^2$

$\nabla f = -2x, -2y, -2z$

$\neq (0,0,0)$ on S^2

- Let $u \subset \mathbb{R}^3$ be open, let $f : U \to \mathbb{R}$ be a function of class C^1, and suppose that the gradient $\nabla_x f \neq 0$ at any point x where $f(x) = 0$. Then

$$\{x \in \mathbb{R}^3 \, | f(x) = 0\}$$

is a topological surface.
- *Reason*: The implicit function theorem. (Will review and prove)

Surfaces as Identification Spaces

Example: Torus

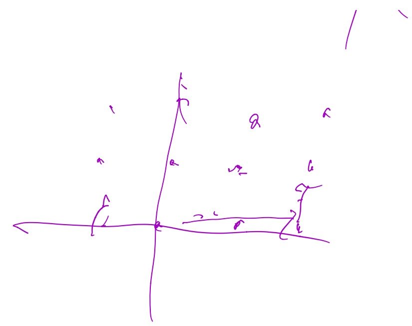
$\mathbb{R}^2 / \mathbb{Z}^2$

$(x, y) \sim (x', y')$

$\Longleftrightarrow \quad x - x' = 2\pi n$

$y - y' = 2\pi m$

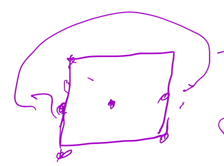
$m, n \in \mathbb{Z}$



every pt in $\mathbb{R}^2$ ~ a pt in $[0,1]\times[0,1]$

$0\le x\le 1$

$0\le y\le 1$

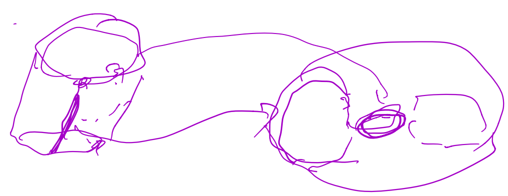


(x,y) $\quad 0<x<1$, $0<y<1$ equiv only to itself

$x=0 \quad (0,y)\sim(1,y)$

$y=0 \quad (x,0)\sim(x,1)$

$(0,0)\sim(0,1)\sim(1,0)\sim(1,1)$



(aka $S^1\times S^1$)

Surface: $[0,1]\times[0,1]/\sim$

$$\boxed{\begin{aligned}(x,0)&\sim(x,1)\\(0,y)&\sim(1,y)\end{aligned}}$$ ← equiv rel gen by

e

(x,y) $\quad 0<x<1$, $0<y<1$ $\quad [(x,y)] = (x,y)$

$(0,0) \sim (0,1)$

$(1,0) \sim (1,1)$

$[(0,0)] = \{(0,0), (1,0), (0,1), (1,1)\}$

Check: Hausdorff, locally homeo to $\mathbb{R}^2$

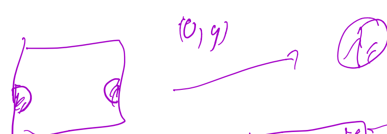
$[(x,y)]$ find nbd homeo $U \subset \mathbb{R}^2$ open.

1) $(x,y) \in$ int of sq. $(I \times I)^\circ$

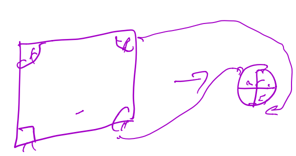
take nbd U of $(x,y) \subset (I \times I)^\circ$

2) $(0,x)$ $\quad x \in I^\circ$

3)

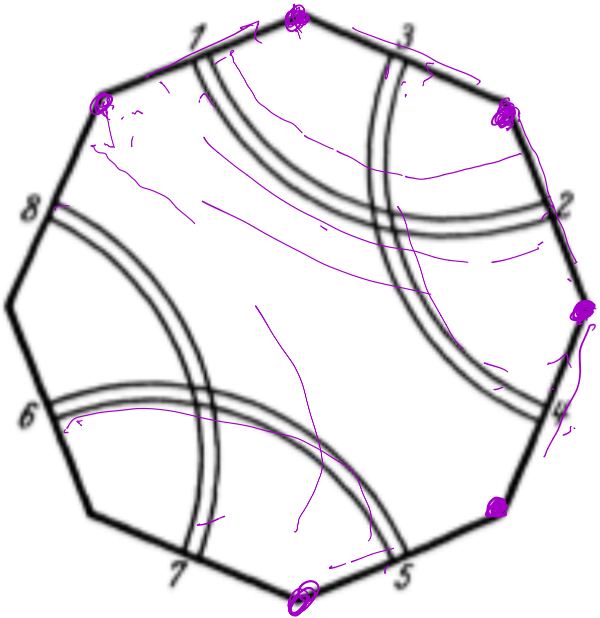


4) $(0,0)$



Surface of Genus Two

octagon



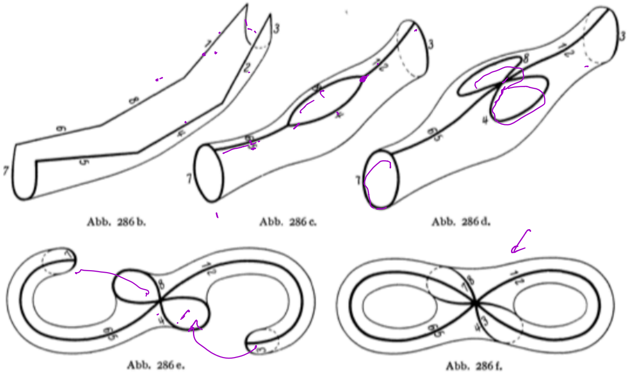


Figure: Surface of Genus Two

?

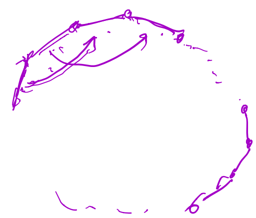
Genus 3

4 8 - - (4g)-gon

Def

Surface S_g of genus g:

regular $4g$-gon.



identify edges

Non-Hausdorff Quotient Spaces

A non-Hausdorff “Surface”

Existence theorems based on connectedness

- The intermediate value theorem.

X conn, $f : X \to \mathbb{R}$ cont

$a < b$

$\exists\, x_1, x_2 \in X$ $\quad f(x_1) = a$

$f(x_2) = b$

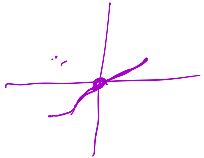
$\Rightarrow \forall c,\ a < c < b$

$\exists\, x \in X$ with $f(x) = c$.

- The implicit function theorem. defined on nbd of $(0,0)$, C^1

$f(x,y)$ $\quad f(0,0) = (0,0)$

$\frac{\partial f}{\partial y}(0,0) \neq 0$



$\exists\ \varphi$ s.t.

$\cancel{f(x, \varphi(x)) = 0}$

s.t. $\{(x,y) \in \text{Nbd} : f(x,y) = 0\}$

$= \{x, \varphi(x)\}$

$f(x,y) = 0$

defines y implicitly as a func x.