Math. Proc. Camb. Phil. Soc. (1999), **127**, 201

Printed in the United Kingdom

$\mathbb{Q}$-Gorenstein splinter rings of characteristic p are F-regular

By ANURAG K. SINGH†

Department of Mathematics, University of Michigan, Ann Arbor, U.S.A.
e-mail: singh6@math.uiuc.edu

(*Received* 5 *May* 1998; *revised* 8 *October* 1998)

1. *Introduction*

A Noetherian integral domain R is said to be a *splinter* if it is a direct summand, as an R-module, of every module-finite extension ring (see [**Ma**]). In the case that R contains the field of rational numbers, it is easily seen that R is splinter if and only if it is a normal ring, but the notion is more subtle for rings of characteristic $p > 0$. It is known that F-regular rings of characteristic p are splinters and Hochster and Huneke showed that the converse is true for locally excellent Gorenstein rings [**HH4**]. In this paper we extend their result by showing that $\mathbb{Q}$-Gorenstein splinters are F-regular. Our main theorem is:

Theorem 1·1. *Let R be a locally excellent $\mathbb{Q}$-Gorenstein integral domain of characteristic $p > 0$. Then R is F-regular if and only if it is a splinter.*

These issues are closely related to the question of whether the *tight closure* I^* of an ideal I of a characteristic p domain agrees with its *plus closure*, i.e. $I^+ = IR^+ \cap R$, where R^+ denotes the integral closure of R in an algebraic closure of its fraction field. We always have the containment $I^+ \subseteq I^*$ and Smith showed that equality holds if I is a parameter ideal in an excellent domain R (see [**Sm1**]). An excellent domain R of characteristic p is splinter if and only if for all ideals I of R, we have $I^+ = I$.

For an excellent local domain R of characteristic p, Hochster and Huneke showed that R^+ is a big Cohen–Macaulay algebra, see [**HH2**]. For further work on R^+ and plus closure see [**Ab**, **AH**]. Our main references for the theory of tight closure are [**HH1**, **HH3**, **HH4**].

Although tight closure is primarily a characteristic p notion, it has strong connections with the study of singularities of algebraic varieties over fields of characteristic zero. For $\mathbb{Q}$-Gorenstein rings essentially of finite type over a field of characteristic zero, it is known that F-regular type is equivalent to log-terminal singularities (see [**Ha**, **Sm2**, **Sm3**, **Wa**]). Consequently our main theorem offers a characterization of log-terminal singularities in characteristic zero, see Corollary 3·3.

2. *Preliminaries*

By the *canonical ideal* of a Cohen–Macaulay normal domain (R, m), we shall mean an ideal of R which is isomorphic to the canonical module of R. We next record some results that we shall use later in our work.

† Present address: Department of Mathematics, University of Illinois, 1409 W. Green Street, Urbana, IL 61801, U.S.A.

LEMMA 2·1. *Let (R, m) be a Cohen–Macaulay local domain with canonical ideal J. Fix a system of parameters $y_1, \ldots, y_d$ for R and let $s \in J$ be an element which represents a socle generator in $J/(y_1, \ldots, y_d)$. Then for $t \in \mathbb{N}$, the element $s(y_1 \cdots y_d)^{t-1}$ is a socle generator in $J/(y_1^t, \ldots, y_d^t)J$. The ideals $I_t = (y_1^t, \ldots y_d^t) J :_R s$ form a family of irreducible ideals which are confinal with the powers of the maximal ideal m of R.*

Proof. See the proof of [**HH4**, theorem 4·6].

LEMMA 2·2. *Let R be a Cohen–Macaulay normal domain with canonical ideal J. Pick $y_1 \neq 0$ in J. Then there exists an element y_2 not in any minimal prime of y_1 and $\gamma \in J$ such that $y_2^i J^{(i)} \subseteq \gamma^i R$ for all positive integers i.*

Proof. This is [**Wi**, lemma 4·3].

LEMMA 2·3. *Let (R, m) be a normal local domain and J an ideal of pure height one, which has order n when regarded as an element of the divisor class group $\mathrm{Cl}(R)$. Then for $0 < i < n$, we have $J^{(i)} J^{(n-i)} \subseteq J^{(n)} m$.*

Proof. Let $J^{(n)} = \alpha R$. Clearly $J^{(i)} J^{(n-i)} \subseteq \alpha R$ and it suffices to show that $J^{(i)} J^{(n-i)} \subsetneq \alpha R$. If $J^{(i)} J^{(n-i)} = \alpha R$, then $J^{(i)}$ is an invertible fractional ideal and so must be a projective R-module. Since R is local, $J^{(i)}$ is a free R-module, but this is a contradiction since $J^{(i)}$ cannot be principal for $0 < i < n$.

Discussion 2·4. Let (R, m) be a $\mathbb{Q}$-Gorenstein Cohen–Macaulay normal local domain, with canonical ideal J. Let n denote the order of J as an element of the divisor class group $\mathrm{Cl}(R)$ and pick $\alpha \in R$ such that $J^{(n)} = \alpha R$. Consider the subring $R[JT, J^{(2)}T^2, \ldots]$ of $R[T]$ and let

$$S = R[JT, J^{(2)}T^2, \ldots]/(\alpha T^n - 1).$$

Note that S has a natural $\mathbb{Z}/n\mathbb{Z}$-grading where $[S]_0 = R$ and for $0 < i < n$ we have $[S]_i = J^{(i)} T^i$. We claim that the ideal

$$\mathfrak{m} = m + JT + J^{(2)}T^2 + \cdots + J^{(n-1)}T^{n-1}$$

is a maximal ideal of S. Since each $J^{(i)}$ is an ideal of R, we need only verify that $J^{(i)} T^i \mathfrak{m} \subseteq \mathfrak{m}$ for $0 < i < n-1$, but this follows from Lemma 2·3. Note furthermore that $\mathfrak{m}^n \subseteq mS$.

3. *The main result*

Proof of Theorem 1·1. The property of being a splinter localizes, as does the property of being $\mathbb{Q}$-Gorenstein. Hence if the splinter ring R is not F-regular, we may localize at a prime ideal $P \in \mathrm{Spec}\, R$ which is minimal with respect to the property that R_P is not F-regular. After a change of notation, we have a splinter (R, m) which has an isolated non F-regular point at the maximal ideal m. This shows that R has an m-primary test ideal. However since R is a splinter it must be F-pure and so the test ideal is precisely the maximal ideal m. Note that by [**Sm1**, theorem 5·1] parameter ideals of R are tightly closed and R is indeed F-rational.

Let $\dim R = d$. Choose a system of parameters for R as follows: first pick a nonzero element $y_1 \in J$. Then, by Lemma 2·2, pick y_2 not in any minimal prime of y_1 such that $y_2^i J^{(i)} \subseteq \gamma^i R$ for a fixed element $\gamma \in J$, for all positive integers i. Extend y_1, y_2 to a full system of parameters $y_1, \ldots, y_d$ for R. Since $y_1 \in J$, there exists $u \in R$ such that $s = uy_1$ is a socle generator in $J/(y_1, \ldots, y_d)J$. Let Y denote the product $y_1 \ldots y_d$.

Consider the family of ideals $\{I_c\}_{c\in\mathbb{N}}$ as in Lemma 2·1. If R is not F-regular, there exists an irreducible ideal $I_c = (y_1^c, \dots, y_d^c)\, J :_R s$ which is not tightly closed, specifically $Y^{c-1} \in I_c^*$. Consequently $sY^{c-1} \in (y_1^c, \dots, y_d^c)\, J^*$ and $sY^{c-1} \in (y_1^c, \dots, y_d^c)\, JS^*$ and so

$$sTY^{c-1} \in (y_1^c, \dots, y_d^c)\, JTS^* \subseteq (y_1^c, \dots, y_d^c)\, S^*.$$

We shall first imitate the proof of [**Sm1**, lemma 5·2] to obtain from this an 'equational condition'. Let $z = sTY^{c-1}$ and $x_i = y_i^c$ for $1 \leqslant i \leqslant d$. We then have $z \in (x_1, \dots, x_d)\, S^*$. Consider the maximal ideal $\mathfrak{m} = m + JT + J^{(2)}T^2 + \dots + J^{(n-1)}T^{n-1}$ of S and the highest local cohomology module

$$H^d_{\mathfrak{m}}(S) = \varinjlim S/(x_1^i, \dots, x_d^i),$$

where the maps in the direct limit system are induced by multiplication by $x_1 \cdots x_d$.

Since the test ideal of R is m, if Q_0 is a power of p greater than n, we have $\mathfrak{m}^{Q_0} z^q \in (x_1^q, \dots, x_d^q)\, S$ for all $q = p^e$.

Let η denote $[z + (x_1, \dots, x_d)\, S]$ viewed as an element of $H^d_{\mathfrak{m}}(S)$ and N be the S-submodule of $H^d_{\mathfrak{m}}(S)$ spanned by all $F^e(\eta)$ where $e \in \mathbb{N}$. Since $H^d_{\mathfrak{m}}(S)$ is an S-module with DCC, there exists e_0 such that the submodules generated by $F^{e_0}(N)$ and $F^{e'}(N)$ agree for all $e' \geqslant e_0$. Hence there exists an equation of the form

$$F^{e_0}(\eta) = a_1 F^{e_1}(\eta) + \dots + a_k F^{e_k}(\eta),$$

with $a_1, \dots, a_k \in S$ and $e_0 < e_1 \leqslant e_2 \leqslant \dots \leqslant e_k$. If some a_i is not a unit, we may use suitably high Frobenius iterations on the equation above and the fact that for $Q_0 \geqslant n$ we have $\mathfrak{m}^{Q_0} F^e(\eta) = 0$ for all $e \in \mathbb{N}$, to replace the above equation by one in which the coefficients which occur are indeed units. Hence we have an equation $F^e(\eta) = a_1 F^{e_1}(\eta) + \dots + a_k F^{e_k}(\eta)$ where $e < e_1 \leqslant e_2 \leqslant \dots \leqslant e_k$ and $a_1, \dots, a_k$ are units. Let $q = p^e$, $q_i = p^{e_i}$ for $1 \leqslant i \leqslant k$ and $X = x_1 \cdots x_d$. Rewriting our equation we have

$$\begin{aligned}[z^q X^{q_k - q} + (x_1^{q_k}, \dots, x_d^{q_k})\, S] = a_1 [z^{q_1} X^{q_k - q_1} + (x_1^{q_k}, \dots, x_d^{q_k})\, S] \\ + \dots + a_k [z^{q_k} + (x_1^{q_k}, \dots, x_d^{q_k})\, S],\end{aligned}$$

i.e. $[z^q X^{q_k - q} - a_1 z^{q_1} X^{q_k - q_1} - \dots - a_k z^{q_k} + (x_1^{q_k}, \dots, x_d^{q_k})\, S] = 0$. Since the ring S may not necessarily be Cohen–Macaulay, we cannot assume that the maps in the direct limit system $\varinjlim S/(x_1^i, \dots, x_d^i)$ are injective. However for a suitable positive integer b we do obtain the equation

$$(zX^{b-1})^Q \in (x_1^{bQ}, \dots, x_d^{bQ}, zX^{bQ-1}, z^p X^{bQ-p}, \dots, z^{Q/p} X^{bQ-Q/p})\, S,$$

where $Q = q_k$. Going back to the earlier notation and setting $t = bc$, we have

$$(sTY^{t-1})^Q \in (y_1^{tQ}, \dots, y_d^{tQ}, sTY^{tQ-1}, (sT)^p\, Y^{tQ-p}, \dots, (sT)^{Q/p}\, Y^{tQ-Q/p})\, S.$$

Note that $1/T = \alpha T^{n-1} \in S$ and, multiplying the above by $1/T^Q$, we get

$$(sY^{t-1})^Q \in \left(y_1^{tQ} \frac{1}{T^Q}, \dots, y_d^{tQ} \frac{1}{T^Q}, sY^{tQ-1} \frac{1}{T^{Q-1}}, s^p Y^{tQ-p} \frac{1}{T^{Q-p}}, \dots, s^{Q/p} Y^{tQ-Q/p} \frac{1}{T^{Q-Q/p}} \right) S.$$

Since $(sY^{t-1})^Q \in [S]_0 = R$, we may intersect the ideal above with R to obtain

$$(sY^{t-1})^Q \in (y_1^{tQ} J^{(Q)}, \dots, y_d^{tQ} J^{(Q)}, sY^{tQ-1} J^{(Q-1)}, s^p Y^{tQ-p} J^{(Q-p)}, \dots, s^{Q/p} Y^{tQ-Q/p} J^{(Q-Q/p)})\, R.$$

Replacing $s = uy_1$ above, we get

$$(uy_1 Y^{t-1})^Q \in (y_1^{tQ} J^{(Q)}, \dots, y_d^{tQ} J^{(Q)}, (uy_1)\, Y^{tQ-1} J^{(Q-1)}, \\ (uy_1)^p\, Y^{tQ-p} J^{(Q-p)}, \dots, (uy_1)^{Q/p}\, Y^{tQ-Q/p} J^{(Q-Q/p)}\, R.$$

Let $Z = Y/y_1 = y_2 \cdots y_d$. We then have

$$(uZ^{t-1})^Q\, y_1^{tQ} \in (y_1^{tQ} J^{(Q)}, y_2^{tQ}, \dots, y_d^{tQ}, uy_1^{tQ} Z^{tQ-1} J^{(Q-1)}, \\ u^p y_1^{tQ} Z^{tQ-p} J^{(Q-p)}, \dots, u^{Q/p} y_1^{tQ} Z^{tQ-Q/p} J^{(Q-Q/p)})\, R.$$

Using the fact that $y_1, \dots, y_d$ are a system of parameters for the Cohen–Macaulay ring R, we get

$$(uZ^{t-1})^Q \in (J^{(Q)}, y_2^{tQ}, \dots, y_d^{tQ}, uZ^{tQ-1} J^{(Q-1)}, u^p Z^{tQ-p} J^{(Q-p)}, \dots, u^{Q/p} Z^{tQ-Q/p} J^{(Q-Q/p)})\, R.$$

Consequently there exists $a \in J^{(Q)}$, $b_i \in R$ and $c_{p^e} \in J^{(Q-Q/p^e)}$ such that

$$(uZ^{t-1})^Q = a + \sum_{i=2}^{d} b_i\, y_i^{tQ} + c_1\, uZ^{tQ-1} + c_p\, u^p Z^{tQ-p} + \cdots + c_{Q/p}\, u^{Q/p} Z^{tQ-Q/p}.$$

For $2 \leqslant i \leqslant d$, consider the following equations in the variables $V_2, \dots, V_d$:

$$V_i^Q = b_i + c_1\, V_i \left(\frac{Z}{y_i}\right)^{tQ-t} + c_p\, V_i^p \left(\frac{Z}{y_i}\right)^{tQ-tp} + \cdots + c_{Q/p}\, V_i^{Q/p} \left(\frac{Z}{y_i}\right)^{tQ-tQ/p}.$$

Since these are monic equations defined over R, there exists a module finite normal extension ring R_1, with solutions v_i of these equations. Working in the ring R_1, let

$$w = uZ^{t-1} - \sum_{i=2}^{d} v_i\, y_i^t.$$

Combining the earlier equations, we have

$$w^Q = a + c_1\, wZ^{tQ-t} + c_p\, w^p Z^{tQ-tp} + \cdots + c_{Q/p}\, w^{Q/p} Z^{tQ-tQ/p}.$$

Multiplying this equation by y_2^Q and using the fact that $y_2^i J^{(i)} \subseteq \gamma^i R$ for all positive integers i, we get

$$(wy_2)^Q = d_0\, \gamma^Q + d_1\, wy_2\, \gamma^{Q-1} + d_p (wy_2)^p\, \gamma^{Q-p} + \cdots + d_{Q/p} (wy_2)^{Q/p}\, \gamma^{Q-Q/p}.$$

The above equation gives an equation by which wy_2/γ is integral over the ring R_1. Since R_1 is normal, we have $wy_2 \in \gamma R_1$. Combining this with $w = uZ^{t-1} - \sum_{i=2}^{d} v_i\, y_i^t$, we have

$$uZ^{t-1} y_2 = wy_2 + \left(\sum_{i=2}^{d} v_i\, y_i^t\right) y_2 \in (J, y_2^{t+1}, y_2\, y_3^t, \dots, y_2\, y_d^t)\, R_1$$

and so

$$uZ^{t-1} y_2 \in (J, y_2^{t+1}, y_2\, y_3^t, \dots, y_2\, y_d^t)^+ = (J, y_2^{t+1}, y_2\, y_3^t, \dots, y_2\, y_d^t)\, R.$$

Since y_2 is not in any minimal prime of J, we get $uZ^{t-1} \in (J, y_2^t, y_3^t, \dots, y_d^t)\, R$. Multiplying this by y_1, we get

$$sZ^{t-1} \in (y_1 J, y_1\, y_2^t, y_1\, y_3^t, \dots, y_1\, y_d^t)\, R \subseteq (y_1, y_2^t, y_3^t, \dots, y_d^t)\, J,$$

but this contradicts the fact that s generates the socle in $J/(y_1, \dots, y_d)\, J$.

Corollary 3·1. *Let (R, m) be an excellent integral domain of dimension two over a field of characteristic $p > 0$. Then R is a splinter if and only if it is F-regular.*

Proof. The hypotheses imply that R is F-rational, and so has a torsion divisor class group by a result of Lipman [**Li**]. Hence R must be $\mathbb{Q}$-Gorenstein.

Definition 3·2. Let $R = K[X_1, \dots, X_n]/I$ be a domain finitely generated over a field K of characteristic zero. We say R is of *splinter type* if there exists a finitely generated $\mathbb{Z}$-algebra $A \subseteq K$ and a finitely generated free A-algebra $R_A = A[X_1, \dots, X_n]/I_A$ such that $R \cong R_A \otimes_A K$, and for all maximal ideals μ in a Zariski dense subset of Spec A, the fibre rings $R_A \otimes_A A/\mu$ (which are rings over fields of characteristic p) are splinter.

Using the equivalence of F-regular type and log-terminal singularities for rings finitely generated over a field of characteristic zero (see [**Ha**, **Sm3**, **Wa**]) we obtain the following corollary:

Corollary 3·3. *Let R be a finitely generated $\mathbb{Q}$-Gorenstein domain over a field of characteristic zero. Then R has log-terminal singularities if and only if it is of splinter type.*

Acknowledgements. It is a pleasure to thank Melvin Hochster for several valuable discussions on tight closure theory.

REFERENCES

[**Ab**] I. Aberbach. Tight closure in F-rational rings. *Nagoya Math. J.* **135** (1994), 43–54.

[**AH**] I. Aberbach and M. Hochster. Finite Tor dimension and failure of coherence in absolute integral closures. *J. Pure Appl. Algebra* **122** (1997), 171–184.

[**Ha**] N. Hara. A characterisation of rational singularities in terms of injectivity of Frobenius maps. *Amer. J. Math.* **120** (1998), 981–996.

[**HH1**] M. Hochster and C. Huneke. Tight closure, invariant theory and the Briançon–Skoda theorem. *J. Amer. Math. Soc.* **3** (1990), 31–116.

[**HH2**] M. Hochster and C. Huneke. Infinite integral extensions and big Cohen–Macaulay algebras. *Ann. of Math.* (2) **135** (1992), 53–89.

[**HH3**] M. Hochster and C. Huneke. F-regularity, test elements and smooth base change. *Trans. Amer. Math. Soc.* **346** (1994), 1–62.

[**HH4**] M. Hochster and C. Huneke. Tight closure of parameter ideals and splitting in module-finite extensions. *J. Algebraic Geom.* **3** (1994), 599–670.

[**Li**] J. Lipman. Rational singularities, with applications to algebraic surfaces and unique factorization. *Inst. Hautes Études Sci. Publ. Math.* **36** (1969), 195–279.

[**Ma**] F. Ma. Splitting in integral extensions, Cohen–Macaulay modules and algebras. *J. Algebra* **116** (1988), 176–195.

[**Sm1**] K. E. Smith. Tight closure of parameter ideals. *Invent. Math.* **115** (1994), 41–60.

[**Sm2**] K. E. Smith. F-rational rings have rational singularities. *Amer. J. Math.* **119** (1997), 159–180.

[**Sm3**] K. E. Smith. Vanishing theorems, singularities and effective bounds in algebraic geometry via prime characteristic local algebra. In *Algebraic geometry – Santa Cruz, 1995*. J. Kollár, R. Lazarsfeld and David R. Morrison (eds.), Proc. Sympos. Pure Math., vol. 62 (Amer. Math. Soc., Providence, RI, 1997), pp. 289–325.

[**Wa**] K.-i. Watanabe. F-purity and F-regularity vs. log-canonical singularities, preprint.

[**Wi**] L. Williams. Uniform stability of kernels of Koszul cohomology indexed by the Frobenius endomorphism. *J. Algebra* **172** (1995), 721–743.