A polynomial identity via differential operators

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Dedicated to Professor Winfried Bruns, on the occasion of his 70th birthday

Abstract We give a new proof of a polynomial identity involving the minors of a matrix, that originated in the study of integer torsion in a local cohomology module.

1 Introduction

Our study of integer torsion in local cohomology modules began in the paper [Si], where we constructed a local cohomology module that has p-torsion for each prime integer p, and also studied the determinantal example $H^3_{I_2}(\mathbb{Z}[X])$ where X is a 2×3 matrix of indeterminates, and I_2 the ideal generated by its size 2 minors. In that paper, we constructed a polynomial identity that shows that the local cohomology module $H^3_{I_2}(\mathbb{Z}[X])$ has no integer torsion; it then follows that this module is a rational vector space. Subsequently, in joint work with Lyubeznik and Walther, we showed that the same holds for all local cohomology modules of the form $H^k_{I_t}(\mathbb{Z}[X])$, where X is a matrix of indeterminates, I_t the ideal generated by its size t minors, and k an integer with $k > \operatorname{height} I_t$, [LSW, Theorem 1.2]. In a related direction, in joint work with Bhatt, Blickle, Lyubeznik, and Zhang, we proved that the local cohomology of a polynomial ring over $\mathbb{Z}$ can have p-torsion for at most finitely many p; we record a special case of [BBLSZ, Theorem 3.1]:

Theorem 1. *Let R be a polynomial ring over the ring of integers, and let $f_1, \dots, f_m$ be elements of R. Let n be a nonnegative integer. Then each prime integer that is a nonzerodivisor on the Koszul cohomology module $H^n(f_1, \dots, f_m; R)$ is also a nonzerodivisor on the local cohomology module $H^n_{(f_1,\dots,f_m)}(R)$.*

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These more general results notwithstanding, a satisfactory proof or conceptual understanding of the polynomial identity from [Si] had previously eluded us; extensive calculations with *Macaulay2* had led us to a conjectured identity, which we were then able to prove using the hypergeometric series algorithms of Petkovšek, Wilf, and Zeilberger [PWZ], as implemented in *Maple*. The purpose of this note is to demonstrate how techniques using differential operators underlying the papers [BBLSZ] and [LSW] provide the “right” proof of the identity, and, indeed, provide a rich source of similar identities.

We remark that there is considerable motivation for studying local cohomology of rings of polynomials with integer coefficients such as $H^k_{I_t}(\mathbb{Z}[X])$: a matrix of indeterminates X specializes to a given matrix of that size over an arbitrary commutative noetherian ring (this is where $\mathbb{Z}$ is crucial), which turns out to be useful in proving vanishing theorems for local cohomology supported at ideals of minors of arbitrary matrices. See [LSW, Theorem 1.1] for these vanishing results, that build upon the work of Bruns and Schwänzl [BS].

2 Preliminary remarks

We summarize some notation and facts. As a reference for Koszul cohomology and local cohomology, we mention [BH]; for more on local cohomology as a $\mathscr{D}$-module, we point the reader towards [Ly1] and [BBLSZ].

Koszul and Čech cohomology

For an element f in a commutative ring R, the Koszul complex $K^\bullet(f;R)$ has a natural map to the Čech complex $C^\bullet(f;R)$ as follows:

$$\begin{array}{cccccccc} K^\bullet(f;R) := & 0 & \longrightarrow & R & \xrightarrow{f} & R & \longrightarrow & 0 \\ \big\downarrow & & & \big\| & & \big\downarrow{\scriptstyle \frac{1}{f}} & & \\ C^\bullet(f;R) := & 0 & \longrightarrow & R & \longrightarrow & R_f & \longrightarrow & 0. \end{array}$$

For a sequence of elements $\boldsymbol{f} = f_1,\dots,f_m$ in R, one similarly obtains

$$K^\bullet(\boldsymbol{f};R) \;:=\; \bigotimes_i K^\bullet(f_i;R) \;\longrightarrow\; \bigotimes_i C^\bullet(f_i;R) \;=:\; C^\bullet(\boldsymbol{f};R),$$

and hence, for each $n \geqslant 0$, an induced map on cohomology modules

$$H^n(\boldsymbol{f};R) \;\longrightarrow\; H^n_{(\boldsymbol{f})}(R). \tag{1}$$

Now suppose R is a polynomial ring over a field $\mathbb{F}$ of characteristic $p > 0$. The Frobenius endomorphism φ of R induces an additive map

$$H^n_{(\boldsymbol{f})}(R) \longrightarrow H^n_{(\boldsymbol{f}^p)}(R) = H^n_{(\boldsymbol{f})}(R),$$

where $\boldsymbol{f}^p = f_1^p, \dots, f_m^p$. Set $R\{\varphi\}$ to be the extension ring of R obtained by adjoining the Frobenius operator, i.e., adjoining a generator φ subject to the relations $\varphi r = r^p \varphi$ for each $r \in R$; see [Ly2, Section 4]. By an $R\{\varphi\}$-module we will mean a left $R\{\varphi\}$-module. The map displayed above gives $H^n_{(\boldsymbol{f})}(R)$ an $R\{\varphi\}$-module structure. It is not hard to see that the image of $H^n(\boldsymbol{f}; R)$ in $H^n_{(\boldsymbol{f})}(R)$ generates the latter as an $R\{\varphi\}$-module; what is much more surprising is a result of Àlvarez, Blickle, and Lyubeznik, [ABL, Corollary 4.4], by which the image of $H^n(\boldsymbol{f}; R)$ in $H^n_{(\boldsymbol{f})}(R)$ generates the latter as a $\mathscr{D}(R, \mathbb{F})$-module; see below for the definition. The result is already notable in the case $m = 1 = n$, where the map (1) takes the form

$$\begin{aligned} H^1(f; R) = R/fR &\longrightarrow R_f/R = H^1_{(f)}(R) \\ [1] &\longmapsto [1/f]. \end{aligned}$$

By [ABL], the element $1/f$ generates R_f as a $\mathscr{D}(R, \mathbb{F})$-module. It is of course evident that $1/f$ generates R_f as an $R\{\varphi\}$-module since the elements $\varphi^e(1/f) = 1/f^{p^e}$ with $e \geqslant 0$ serve as R-module generators for R_f. See [BDV] for an algorithm to explicitly construct a differential operator δ with $\delta(1/f) = 1/f^{p^e}$, along with a *Macaulay2* implementation.

Differential operators

Let A be a commutative ring, and x an indeterminate; set $R = A[x]$. The divided power partial differential operator

$$\frac{1}{k!} \frac{\partial^k}{\partial x^k}$$

is the A-linear endomorphism of R with

$$\frac{1}{k!} \frac{\partial^k}{\partial x^k}(x^m) = \binom{m}{k} x^{m-k} \qquad \text{for } m \geqslant 0,$$

where we use the convention that the binomial coefficient $\binom{m}{k}$ vanishes if $m < k$. Note that

$$\frac{1}{r!} \frac{\partial^r}{\partial x^r} \cdot \frac{1}{s!} \frac{\partial^s}{\partial x^s} = \binom{r+s}{r} \frac{1}{(r+s)!} \frac{\partial^{r+s}}{\partial x^{r+s}}.$$

For the purposes of this paper, if R is a polynomial ring over A in the indeterminates $x_1, \dots, x_d$, we define the ring of A-linear differential operators on R, denoted $\mathscr{D}(R,A)$, to be the free R-module with basis

$$\frac{1}{k_1!}\frac{\partial^{k_1}}{\partial x_1^{k_1}} \cdot \cdots \cdot \frac{1}{k_d!}\frac{\partial^{k_d}}{\partial x_d^{k_d}} \qquad \text{for } k_i \geqslant 0,$$

with the ring structure coming from composition. This is consistent with more general definitions; see [Gr, 16.11]. By a $\mathscr{D}(R,A)$-module, we will mean a *left* $\mathscr{D}(R,A)$-module; the ring R has a natural $\mathscr{D}(R,A)$-module structure, as do localizations of R. For a sequence of elements $\boldsymbol{f}$ in R, the Čech complex $C^\bullet(\boldsymbol{f};R)$ is a complex of $\mathscr{D}(R,A)$-modules, and hence so are its cohomology modules $H^n_{(\boldsymbol{f})}(R)$. Note that for $m \geqslant 1$, one has

$$\frac{1}{k!}\frac{\partial^k}{\partial x^k}\left(\frac{1}{x^m}\right) = (-1)^k\binom{m+k-1}{k}\frac{1}{x^{m+k}}.$$

We also recall the Leibniz rule, which states that

$$\frac{1}{k!}\frac{\partial^k}{\partial x^k}(fg) = \sum_{i+j=k}\frac{1}{i!}\frac{\partial^i}{\partial x^i}(f)\,\frac{1}{j!}\frac{\partial^j}{\partial x^j}(g).$$

3 The identity

Let R be the ring of polynomials with integer coefficients in the indeterminates

$$\begin{pmatrix} u & v & w \\ x & y & z \end{pmatrix}.$$

The ideal I generated by the size 2 minors of the above matrix has height 2; our interest is in proving that the local cohomology module $H^3_I(R)$ is a rational vector space. We label the minors as $\Delta_1 = vz - wy$, $\Delta_2 = wx - uz$, and $\Delta_3 = uy - vx$. Fix a prime integer p, and consider the exact sequence

$$0 \longrightarrow R \xrightarrow{\ p\ } R \longrightarrow \overline{R} \longrightarrow 0,$$

where $\overline{R} = R/pR$. This induces an exact sequence of local cohomology modules

$$\longrightarrow H^2_I(R) \xrightarrow{\ \pi\ } H^2_I(\overline{R}) \longrightarrow H^3_I(R) \xrightarrow{\ p\ } H^3_I(R) \longrightarrow H^3_I(\overline{R}) \longrightarrow 0.$$

The ring $\overline{R}/I\overline{R}$ is Cohen-Macaulay of dimension 4, so [PS, Proposition III.4.1] implies that $H^3_I(\overline{R}) = 0$. As p is arbitrary, it follows that $H^3_I(R)$ is a divisible abelian group. To prove that it is a rational vector space, one needs to show that multiplication by p on $H^3_I(R)$ is injective, equivalently that π is surjective. We first prove this using the identity (2) below, and then proceed with the proof of the identity.

For each $k \geqslant 0$, one has

$$\sum_{i,j\geqslant 0}\binom{k}{i+j}\binom{k+i}{k}\binom{k+j}{k}\frac{(-wx)^i(vx)^j u^{k+1}}{\Delta_2^{k+1+i}\Delta_3^{k+1+j}}$$
$$+\sum_{i,j\geqslant 0}\binom{k}{i+j}\binom{k+i}{k}\binom{k+j}{k}\frac{(-uy)^i(wy)^j v^{k+1}}{\Delta_3^{k+1+i}\Delta_1^{k+1+j}}$$
$$+\sum_{i,j\geqslant 0}\binom{k}{i+j}\binom{k+i}{k}\binom{k+j}{k}\frac{(-vz)^i(uz)^j w^{k+1}}{\Delta_1^{k+1+i}\Delta_2^{k+1+j}} = 0. \quad (2)$$

Since the binomial coefficient $\binom{k}{i+j}$ vanishes if i or j exceeds k, this equation may be rewritten as an identity in the polynomial ring $\mathbb{Z}[u,v,w,x,y,z]$ after multiplying by $(\Delta_1\Delta_2\Delta_3)^{2k+1}$.

Computing $H_I^2(R)$ as the cohomology of the Čech complex $C^\bullet(\Delta_1,\Delta_2,\Delta_3;R)$, equation (2) gives a 2-cocycle in

$$C^2(\Delta_1,\Delta_2,\Delta_3;R) = R_{\Delta_1\Delta_2}\oplus R_{\Delta_1\Delta_3}\oplus R_{\Delta_2\Delta_3};$$

we denote the cohomology class of this cocycle in $H_I^2(R)$ by η_k. When $k = p^e-1$, one has

$$\binom{k}{i+j}\binom{k+i}{k}\binom{k+j}{k}\equiv 0 \mod p \quad \text{for } (i,j)\neq(0,0),$$

so (2) reduces modulo p to

$$\frac{u^{p^e}}{\Delta_2^{p^e}\Delta_3^{p^e}}+\frac{v^{p^e}}{\Delta_3^{p^e}\Delta_1^{p^e}}+\frac{w^{p^e}}{\Delta_1^{p^e}\Delta_2^{p^e}}\equiv 0 \mod p,$$

and the cohomology class η_{p^e-1} has image

$$\pi(\eta_{p^e-1}) = \left[\left(\frac{w^{p^e}}{\Delta_1^{p^e}\Delta_2^{p^e}},\frac{-v^{p^e}}{\Delta_1^{p^e}\Delta_3^{p^e}},\frac{u^{p^e}}{\Delta_2^{p^e}\Delta_3^{p^e}}\right)\right] \qquad \text{in } H_I^2(\overline{R}).$$

Since $\overline{R}$ is a regular ring of positive characteristic, $H_I^2(\overline{R})$ is generated as an $\overline{R}\{\varphi\}$-module by the image of

$$H^2(\Delta_1,\Delta_2,\Delta_3;\overline{R}) \longrightarrow H_I^2(\overline{R}).$$

The Koszul cohomology module $H^2(\Delta_1,\Delta_2,\Delta_3;\overline{R})$ is readily seen to be generated, as an $\overline{R}$-module, by elements corresponding to the relations

$$u\Delta_1+v\Delta_2+w\Delta_3=0 \qquad \text{and} \qquad x\Delta_1+y\Delta_2+z\Delta_3=0.$$

These two generators of $H^2(\Delta_1,\Delta_2,\Delta_3;\overline{R})$ map, respectively, to

$$\alpha := \left[\left(\frac{w}{\Delta_1\Delta_2},\frac{-v}{\Delta_1\Delta_3},\frac{u}{\Delta_2\Delta_3}\right)\right] \quad \text{and} \quad \beta := \left[\left(\frac{z}{\Delta_1\Delta_2},\frac{-y}{\Delta_1\Delta_3},\frac{x}{\Delta_2\Delta_3}\right)\right]$$

in $H^2_I(\overline{R})$. Thus, $H^2_I(\overline{R})$ is generated over $\overline{R}$ by $\varphi^e(\alpha)$ and $\varphi^e(\beta)$ for $e \geqslant 0$. But

$$\varphi^e(\alpha) = \pi(\eta_{p^e-1})$$

is in the image of π, and hence so is $\varphi^e(\beta)$ by symmetry. Thus, π is surjective.

The proof of the identity

We start by observing that $C^2(\Delta_1, \Delta_2, \Delta_3; R)$ is a $\mathscr{D}(R, \mathbb{Z})$-module. The element

$$\left(\frac{w}{\Delta_1 \Delta_2}, \frac{-v}{\Delta_1 \Delta_3}, \frac{u}{\Delta_2 \Delta_3} \right)$$

is a 2-cocycle in $C^2(\Delta_1, \Delta_2, \Delta_3; R)$ since

$$\frac{w}{\Delta_1 \Delta_2} + \frac{v}{\Delta_1 \Delta_3} + \frac{u}{\Delta_2 \Delta_3} = 0. \tag{3}$$

We claim that the identity (2) is simply the differential operator

$$D = \frac{1}{k!} \frac{\partial^k}{\partial u^k} \cdot \frac{1}{k!} \frac{\partial^k}{\partial y^k} \cdot \frac{1}{k!} \frac{\partial^k}{\partial z^k}$$

applied termwise to (3); we first explain the choice of this operator: set $k = p^e - 1$, and consider $\overline{D} = D \mod p$ as an element of

$$\mathscr{D}(R, \mathbb{Z})/p\mathscr{D}(R, \mathbb{Z}) = \mathscr{D}(R/pR, \mathbb{Z}/p\mathbb{Z}).$$

It is an elementary verification that

$$\begin{aligned} \overline{D}(u\Delta_2^{p^e-1}\Delta_3^{p^e-1}) &\equiv u^{p^e} \\ \overline{D}(v\Delta_3^{p^e-1}\Delta_1^{p^e-1}) &\equiv v^{p^e} \qquad \mod p. \\ \overline{D}(w\Delta_1^{p^e-1}\Delta_2^{p^e-1}) &\equiv w^{p^e} \end{aligned}$$

Since $k < p^e$, the differential operator $\overline{D}$ is $\overline{R}^{p^e}$-linear; dividing the above equations by $\Delta_2^{p^e}\Delta_3^{p^e}$, $\Delta_3^{p^e}\Delta_1^{p^e}$, and $\Delta_1^{p^e}\Delta_2^{p^e}$ respectively, we obtain

$$\overline{D}\left(\frac{w}{\Delta_1 \Delta_2}, \frac{-v}{\Delta_1 \Delta_3}, \frac{u}{\Delta_2 \Delta_3} \right) \equiv \left(\frac{w^{p^e}}{\Delta_1^{p^e}\Delta_2^{p^e}}, \frac{-v^{p^e}}{\Delta_1^{p^e}\Delta_3^{p^e}}, \frac{u^{p^e}}{\Delta_2^{p^e}\Delta_3^{p^e}} \right) \mod p,$$

which maps to the desired cohomology class $\varphi^e(\alpha)$ in $H^2_I(\overline{R})$. Of course, the operator D is not unique in this regard.

Using elementary properties of differential operators recorded in §2, we have

$$\begin{aligned}
D\left(\frac{v}{\Delta_3\Delta_1}\right) &= \frac{1}{k!}\frac{\partial^k}{\partial u^k}\cdot\frac{1}{k!}\frac{\partial^k}{\partial y^k}\cdot\frac{1}{k!}\frac{\partial^k}{\partial z^k}\ \left[\frac{v}{(uy-vx)(vz-wy)}\right]\\
&= \frac{1}{k!}\frac{\partial^k}{\partial u^k}\cdot\frac{1}{k!}\frac{\partial^k}{\partial y^k}\ \left[\frac{v(-v)^k}{(uy-vx)(vz-wy)^{k+1}}\right]\\
&= \frac{1}{k!}\frac{\partial^k}{\partial y^k}\ \left[\frac{v(-v)^k(-y)^k}{(uy-vx)^{k+1}(vz-wy)^{k+1}}\right]\\
&= v^{k+1}\frac{1}{k!}\frac{\partial^k}{\partial y^k}\ \left[\frac{y^k}{(uy-vx)^{k+1}(vz-wy)^{k+1}}\right]\\
&= v^{k+1}\sum_{i,j}\left[\frac{1}{i!}\frac{\partial^i}{\partial y^i}\frac{1}{(uy-vx)^{k+1}}\right]\left[\frac{1}{j!}\frac{\partial^j}{\partial y^j}\frac{1}{(vz-wy)^{k+1}}\right]\left[\frac{1}{(k-i-j)!}\frac{\partial^{k-i-j}}{\partial y^{k-i-j}}y^k\right]\\
&= v^{k+1}\sum_{i,j}\binom{k+i}{i}\frac{(-u)^i}{(uy-vx)^{k+1+i}}\binom{k+j}{j}\frac{w^j}{(vz-wy)^{k+1+j}}\binom{k}{i+j}y^{i+j}\\
&= v^{k+1}\sum_{i,j}\binom{k+i}{i}\binom{k+j}{j}\binom{k}{i+j}\frac{(-uy)^i(wy)^j}{\Delta_3^{k+1+i}\Delta_1^{k+1+j}}.
\end{aligned}$$

A similar calculation shows that

$$D\left(\frac{w}{\Delta_1\Delta_2}\right) = w^{k+1}\sum_{i,j}\binom{k+i}{i}\binom{k+j}{j}\binom{k}{i+j}\frac{(-vz)^i(uz)^j}{\Delta_1^{k+1+i}\Delta_2^{k+1+j}}.$$

It remains to evaluate $D\left(\frac{u}{\Delta_2\Delta_3}\right)$; we reduce this to the previous calculation as follows. First note that the differential operators $\frac{\partial}{\partial u}\cdot\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial v}\cdot\frac{\partial}{\partial x}$ commute; it is readily checked that they agree on $\frac{u}{\Delta_2\Delta_3}$. Consequently the operators

$$\frac{1}{k!}\frac{\partial^k}{\partial u^k}\cdot\frac{1}{k!}\frac{\partial^k}{\partial y^k}\cdot\frac{1}{k!}\frac{\partial^k}{\partial z^k} \quad \text{and} \quad \frac{1}{k!}\frac{\partial^k}{\partial v^k}\cdot\frac{1}{k!}\frac{\partial^k}{\partial z^k}\cdot\frac{1}{k!}\frac{\partial^k}{\partial x^k}$$

agree on $\frac{u}{\Delta_2\Delta_3}$ as well. But then

$$D\left(\frac{u}{\Delta_2\Delta_3}\right) = \frac{1}{k!}\frac{\partial^k}{\partial v^k}\cdot\frac{1}{k!}\frac{\partial^k}{\partial z^k}\cdot\frac{1}{k!}\frac{\partial^k}{\partial x^k}\ \left[\frac{u}{(wx-uz)(uy-vx)}\right]$$

which, using the previous calculation and symmetry, equals

$$u^{k+1}\sum_{i,j}\binom{k+i}{i}\binom{k+j}{j}\binom{k}{i+j}\frac{(-wx)^i(vx)^j}{\Delta_2^{k+1+i}\Delta_3^{k+1+j}}.$$

Identities in general

Suppose $\boldsymbol{f} = f_1, \dots, f_m$ are elements of a polynomial ring R over $\mathbb{Z}$, and $g_1, \dots, g_m$ are elements of R such that

$$g_1 f_1 + \cdots + g_m f_m = 0.$$

Then, for each prime integer p and $e \geqslant 0$, the Frobenius map on $\overline{R} = R/pR$ gives

$$g_1^{p^e} f_1^{p^e} + \cdots + g_m^{p^e} f_m^{p^e} \equiv 0 \mod p. \tag{4}$$

Now suppose p is a nonzerodivisor on the Koszul cohomology module $H^m(\boldsymbol{f}; R)$. Then Theorem 1 implies that (4) *lifts* to an equation

$$G_1 f_1^N + \cdots + G_m f_m^N = 0 \tag{5}$$

in R in the sense that the cohomology class corresponding to (5) in $H^{m-1}_{(\boldsymbol{f})}(R)$ maps to the cohomology class corresponding to (4) in $H^{m-1}_{(\boldsymbol{f})}(\overline{R})$.

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