IN CLASS WORKSHEET #2, GETTING A FEEL FOR SERRE'S CONDITION $\mathbf{S}_n$

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Our goal is to prove the following result.

Proposition 1. *Suppose that* $(R, \mathfrak{m})$ *is a* d*-dimensional* $\mathbf{S}_n$ $(n \leq d)$ *Noetherian equidimensional*[1] *local ring with normalized dualizing complex* $\omega_R^\bullet$*. Then* $\dim \operatorname{Supp} h^{-i}(\omega_R^\bullet) \leq i - n$ *for* $i < d$*.*

To prove this, you may use the following facts:

- Local duality, $\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(M, \omega_R^\bullet), E) \cong \mathbf{R}\Gamma_{\mathfrak{m}}(M)$.
- Since $h^{-i}(\omega_R^\bullet)$ is a finitely generated R-module, $\operatorname{Supp} h^{-i}(\omega_R^\bullet) = V(\operatorname{Ann}_R(h^{-i}(\omega_R^\bullet)))$
- Since R is equidimensional, $\omega_R = h^{-d}(\omega_R^\bullet)$ has support equal to $\operatorname{Spec} R$.
- Rings with dualizing complexes are catenary, and so every maximal chain of primes between two fixed primes has the same length.

1. For any prime Q of R, show that $d = \dim R_Q + \dim R_Q$. Give an example where this does not hold for a *non*-equidimensional (but otherwise nice) ring.

Now we prove the following weak version of the proposition.

2. Suppose that $(R, \mathfrak{m})$ is a d-dimensional Notherian ring with normalized dualizing complex $\omega_R^\bullet$. Show that $h^i(\omega_R^\bullet) = 0$ for $i > 0$.

[1]This means for each minimal prime $\mathfrak{q}$ of R, $\dim R_{\mathfrak{q}} = \dim R$.

3. Suppose that $Q \in \operatorname{Spec} R$ is a prime ideal such that $\dim R_Q = c$. Find a normalized dualizing complex for R_Q.

4. Suppose that $(R, \mathfrak{m})$ is $\mathbf{S}_n$ $(n \leq d)$ show that $h^{-i}(\omega_R^{\bullet}) = 0$ for $i < n$.

5. Prove the proposition from the first page.

Hint: If $\dim \operatorname{Supp} h^{-i}(\omega_R^{\bullet}) > i - n$ for some $i < d$ choose an appropriate prime to localize at, then keep track of the *normalized* dualizing complex.

Let's do some other applications of dualizing complexes. Recall that a local ring $(R, \mathfrak{m})$ is called F-injective if for every $i \geq 0$, the induced Frobenius action

$$H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(F_* R)$$

is injective. A Noetherian ring in general is called F-injective if for each maximal $\mathfrak{m} \in \operatorname{Spec} R$, $R_{\mathfrak{m}}$ is F-injective.

6. Suppose that $(R, \mathfrak{m})$ is an F-finite Noetherian local ring. Show that for each $Q \in \operatorname{Spec} R$, R_Q is F-injective.

7. Suppose that R is an F-finite Noetherian ring with dualizing complex $\omega_R^{\bullet}$. Show that R is F-injective if and only if the induced map $F_* \omega_{R^{\bullet}} \cong \omega^{\bullet}_{F_* R} \to \omega^{\bullet}_R$ surjects on all cohomology (*i.e.* , $F_* h^i(\omega^{\bullet}_R) \to h^i \omega^{\bullet}_R$ for all i).