IN CLASS WORKSHEET #3, DIVISORS AND REFLEXIVE SHEAVES

MARCH 31ST, 2017

Throughout this worksheet, you should assume that *all rings are Noetherian normal domains.*

1. Show that for any finitely generated R-module M, $\mathrm{Hom}_R(M,R)$ is reflexive.

Solution: Set $N = \mathrm{Hom}_R(M,R)$, and note it is torsion free. Consider the canonical map $i : N \longrightarrow N^{\vee\vee} =: \mathrm{Hom}(\mathrm{Hom}(N,R),R)$, which is a map which is generically an isomorphism between finitely generated torsion free modules. The map i is thus injective. On the other hand, we have the map $k : M \longrightarrow M^{\vee\vee} = \mathrm{Hom}_R(\mathrm{Hom}_R(M,R),R)$ (sending m to the map which sends $\alpha \in \mathrm{Hom}_R(M,R)$ to $\alpha(m)$), applying $\mathrm{Hom}_R(\_\_,R)$ gives us $j : N^{|vee\vee} = \mathrm{Hom}_R(M^{\vee\vee},R) \longrightarrow \mathrm{Hom}_R(M,R) = N$. j is also injective and so generically an isomorphism. Thus we have a composition

$$N \xrightarrow{i} N^{\vee\vee} \xrightarrow{j} N$$

which is also generically an isomorphism and hence injective. Let's show it's actually surjective too. For that we interpret it.

Fix $\phi \in N = \mathrm{Hom}_R(M,N)$. Then for any $\alpha \in \mathrm{Hom}_R(N,R)$, $i(\phi)(\alpha) = \alpha(\phi)$. On the other hand, given any $\psi \in N^{\vee\vee} = \mathrm{Hom}_R(M^{\vee\vee},R)$, we see that $j(\psi)(m) = \psi(k(m))$. Putting this together, for $\phi \in N$,

$$j(i(\phi))(m) = i(\phi)(k(m)) = (k(m))(\phi) = \phi(m).$$

In particular, $j(i(\phi)) = \phi$. It follows that $j \circ i$ is also surjective. Thus j is both surjective and injective and hence an isomorphism.

2. Show that the canonical map $M \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(M,R),R) =: M^{\vee\vee}$ is injective if and only if M is torsion free.

Solution: Since $M^{\vee\vee}$ is torsion free, if the map is injective, M is a subset of a torsion free module and so also torsion free. Conversely suppose that M is torsion free, then $M \longrightarrow M^{\vee\vee}$ is generically an isomorphism (since M localized at the generic point of R, $M_{K(R)} = M \otimes_R K(R)$ is a $K(R)$-vector space). Thus we have the diagram

$$\begin{array}{ccc} M & \longrightarrow & M^{\vee\vee} \\ \big\downarrow & & \big\downarrow \\ M_{K(R)} & \xrightarrow{\sim} & (M^{\vee\vee})_{K(R)} \end{array}$$

A quick diagram trace implies that $M \longrightarrow M^{\vee\vee}$ is injective.

3. Show directly that a finitely generated $\mathbf{S}_2$-module M is reflexive (you may assume that reflexive modules are $\mathbf{S}_2$).

Hint: The fact that it is $\mathbf{S}_1$ implies it is torsion free and so we have a short exact sequence

$$0 \longrightarrow M \longrightarrow M^{\vee\vee} \longrightarrow C \longrightarrow 0.$$

First show that C is zero in codimension 1 on R (localize, then R is a PID, and so M is a nice direct sum). Next use the $\mathbf{S}_2$ hypothesis.

4. Suppose $\Phi \in \mathrm{Hom}_R(F^e_* R, R)$ generates the module (as an $F^e_* R$-module), then show *carefully* that $\Delta_\Phi = 0 = D_\Phi$. Furthermore, if we write $\phi(F^e_* \_\_) = \psi(F^e_*(r \cdot \_\_))$ for some $r \in R$, then $D_\phi = D_\psi + \mathrm{div}_R(r)$ and so $\Delta_\phi = \Delta_\psi + \frac{1}{p^e-1}\,\mathrm{div}_R(r)$.

Hint: We sketched why this should be true in class, but didn't prove it. You get to prove it carefully.

5. In the ring $R = k[x^3, x^2y, xy^2, y^3]$, you may pick your canonical divisor K_R to be $\langle x^2y, xy^2, y^3\rangle$. Verify that K_R is not Cartier but that $3K_R \sim 0$ and in particular, $3K_R$ is Cartier.

6. Consider the ring R from the previous problem and set $\operatorname{char} k = 7$. Suppose I tell you that $R \cong k[a, b, c, d]/\langle c^2 - bd, bc - ad, b^2 - ac\rangle = S/J$ and that $J^{[7]} : J$ has 4 generators, which are

$$= \left\{ \begin{array}{c} g_1 = c^{14} - b^7d^7, \\ g_2 = b^7c^7 - a^7d^7, \\ g_3 = b^{14} - a^7c^7, \\ g_4 = a^2b^6c^{12} + a^3b^4c^{13} + a^3b^5c^{11}d + a^4b^3c^{12}d + a^5bc^{13}d + b^{12}c^6d^2 \\ +a^3b^6c^9d^2 + a^4b^4c^{10}d^2 + a^5b^2c^{11}d^2 + a^6c^{12}d^2 + b^{13}c^4d^3 \\ +ab^{11}c^5d^3 + a^2b^9c^6d^3 + a^4b^5c^8d^3 + a^5b^3c^9d^3 + a^6bc^{10}d^3 + ab^{12}c^3d^4 \\ +a^2b^{10}c^4d^4 + a^3b^8c^5d^4 + a^4b^6c^6d^4 + a^5b^4c^7d^4 + a^6b^2c^8d^4 + a^7c^9d^4 \\ +ab^{13}cd^5 + a^2b^{11}c^2d^5 + a^3b^9c^3d^5 + a^4b^7c^4d^5 + a^5b^5c^5d^5 + a^6b^3c^6d^5 \\ +a^7bc^7d^5 + a^2b^{12}d^6 + a^3b^{10}cd^6 + a^4b^8c^2d^6 + a^5b^6c^3d^6 + a^6b^4c^4d^6 \\ +a^7b^2c^5d^6 + a^8c^6d^6 + a^4b^9d^7 + a^5b^7cd^7 + a^6b^5c^2d^7 + a^7b^3c^3d^7 + a^8bc^4d^7 \\ +a^6b^6d^8 + a^7b^4cd^8 + a^8b^2c^2d^8 + a^9c^3d^8 + a^8b^3d^9 + a^9bcd^9 + a^{10}d^{10} \end{array} \right\}$$

Verify that R is $\mathbb{Q}$-Gorenstein and that $6K_R \sim 0$. Furthermore, identify an homomorphism $\phi \in \operatorname{Hom}_S(F_*S, S)$ such that $\phi(F_*J) \subseteq J$ and that the induced map on $\phi_{S/J}$ generates $\operatorname{Hom}_R(F_*R, R)$.