IN CLASS WORKSHEET #1, AN INTERPRETATION OF THE OCTAHEDRAL AXIOM

JANUARY 27TH, 2017

You may have seen the following about R-modules.

Suppose that $g : B \longrightarrow D$ and $h : C \longrightarrow D$ are R-module homomorphisms. Then the pullback of $\{B \longrightarrow D \leftarrow C\}$ is the set of pairs $(b, c) \in B \oplus C$ such that $g(b) = h(c)$. In other words, it is the kernel of

$$B \oplus C \xrightarrow{g-h} D.$$

Now suppose that we have an exact sequence

$$0 \longrightarrow A \xrightarrow{\alpha \oplus \beta} B \oplus C \xrightarrow{g-h} D$$

In particular, suppose A is isomorphic to the pullback/limit of $\{B \longrightarrow D \leftarrow C\}$ in the category of modules.

1. With notation above, show that if we form the two exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_1 & \xrightarrow{i} & A & \xrightarrow{\alpha} & B \\ & & \Big\downarrow{\scriptstyle \gamma} & & \Big\downarrow{\scriptstyle \beta} & & \Big\downarrow{\scriptstyle g} \\ 0 & \longrightarrow & K_2 & \xrightarrow[j]{} & C & \xrightarrow[h]{} & D \end{array}$$

that the induced map γ is an isomorphism.

2. If you are not too tired of element chasing, suppose that

$$A \xrightarrow{\alpha \oplus \beta} B \oplus C \xrightarrow{g-h} D \longrightarrow 0$$

is exact. In other words, suppose that D is the pushout/colimit of $\{C \leftarrow A \rightarrow B\}$ in the category of modules.

Prove that if we form two exact sequences

$$\begin{array}{ccccccc} A & \xrightarrow{\alpha} & B & \xrightarrow{p} & Q_1 & \longrightarrow & 0 \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle g} & & \vdots{\scriptstyle \delta} & & \\ C & \xrightarrow[h]{} & D & \xrightarrow[q]{} & Q_2 & \longrightarrow & 0 \end{array}$$

and A is isomorphic to the pullback as above, that δ is an isomorphism.

Remark: Various converse statements are also possible and probably even more standard, ie if certain diagrams like the above are exact then various squares are pullbacks and pushouts respectively. We'll just go this way...

3. Let $\mathscr{C}$ be a triangulated category. Suppose that we have objects A, B, C, D and arrows $s : A \to B$, $t : A \to C$, $u : B \to D$, $v : C \to D$ such that

$$A \xrightarrow{s,-t} B \oplus C \xrightarrow{u+v} D \xrightarrow{r} T(A)$$

is an exact triangle. Show that the following holds: If $A \xrightarrow{s} B \xrightarrow{\phi} M \xrightarrow{\psi} T(A)$ is an exact triangle, so is $C \xrightarrow{v} D \xrightarrow{f} M \xrightarrow{g} T(C)$. Furthermore show that $M \xrightarrow{g} T(C)$ factors through $M \xrightarrow{\psi} T(A)$ and $B \xrightarrow{\phi} M$ factors through $B \xrightarrow{u} D$. In particular, this means we have a map of exact triangles

$$\begin{array}{ccccccc} A & \xrightarrow{s} & B & \xrightarrow{\phi} & M & \xrightarrow{\psi} & \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle u} & & \| & & \\ C & \xrightarrow{v} & D & \xrightarrow{f} & M & \xrightarrow{g} & . \end{array}$$

Hint: Let π_1 and π_2 be the projections from $B \oplus C$ to B and C respectively. Let ρ_1 and ρ_2 be the inclusions of B and C into $B \oplus C$. We have the following exact triangle as well

$$C \xrightarrow{\rho_2} B \oplus C \xrightarrow{\pi_1} B \xrightarrow{0} T(C)$$

Rotate this triangle appropriately.