EXERCISES FOR CHARACTERISTIC p COMMUTATIVE ALGEBRA FEBRUARY 27RD, 2017

DUE, FRIDAY MARCH 10TH, 2017

(1) For R a Noetherian ring, $I \subseteq R$ an ideal and $U = \operatorname{Spec} R \setminus V(I)$ we defined for any R-module M

$$\Gamma(U, M) = \ker\Big(\bigoplus_j M_{f_j} \longrightarrow \bigoplus_{a<b} M_{f_a f_b}\Big)$$

where $I = \langle f_1, \ldots, f_m \rangle$. Prove that $\Gamma(U, M)$ is independent of the choice of generators of I.

(2) Suppose that R is a Noetherian ring. Recall that an object $\omega^\bullet \in D^b_{f.g.}(R)$ is called a *dualizing complex for* R if the following two conditions are satisfied.

(a) $\omega^\bullet$ has finite injective dimension (is quasi-isomorphic to a bounded complex of injectives) and,

(b) The functor $\mathbf{D}(\_\_) = \mathbf{R}\operatorname{Hom}_R(\_\_, \omega^\bullet)$ has the property that the canonical map $C^\bullet \longrightarrow \mathbf{D}(\mathbf{D}(C^\bullet))$ is an isomorphism for all $C^\bullet \in D^b_{f.g.}(R)$.

(b') Or equivalently to (b), $R \cong \mathbf{R}\operatorname{Hom}_R(\omega^\bullet, \omega^\bullet)$.

Prove that (b') implies (b) above.

Hint: If you are stuck you may use the following formula (or prove it if you are feeling ambitious). For $A^\bullet, B^\bullet, C^\bullet \in D^b_{f.g.}(R)$ with $C^\bullet$ quasi-isomorphic to a finite complex of injectives, we have

$$A^\bullet \otimes^{\mathbf{L}}_R \mathbf{R}\operatorname{Hom}_R(B^\bullet, C^\bullet) \simeq_{\text{qis}} \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(A^\bullet, B^\bullet), C^\bullet)$$

(3) Suppose that $(R, \mathfrak{m})$ is an F-finite Noetherian local ring of characteristic $p > 0$ with a dualizing complex. Then $\mathbf{R}\operatorname{Hom}_R(F^e_* R, \omega^\bullet_R) =: \omega^\bullet_{F^e_* R}$ is a dualizing complex for $F^e_* R$. Furthermore, $F^e_* \omega^\bullet_R \simeq_{\text{qis}} \omega^\bullet_{F^e_* R}$.

Hint: The fact that it is a dualizing complex was worked out in class. Dualizing complexes are unique up to shifting and twisting by rank-one projectives. Use the fact that it is local to handle the rank-1 projective case. Then use the that the localization of a dualizing complex is a dualizing complex to handle the shift (localize at a minimal prime).

(4) Show that the formation of R^{N} (the normalization of R) commutes with localization, in particular if $W \subseteq R$ is a multiplicative set then $(W^{-1}R)^{\mathrm{N}} = W^{-1}(R^{\mathrm{N}})$.

(5) Suppose that $R = k[x]$ ($\operatorname{char} k \neq 2$) and consider the three points $x = -1, x = 0, x = 1$. Glue these three points together and find a presentation of the resulting non-normal ring. Give a geometric justification about why the Spec of it cannot be embedded as a closed subscheme of $\mathbb{A}^2 = \operatorname{Spec} k[u, v]$.

(6) Fix R an F-finite ring of characteristic $p > 0$, $\mathfrak{a} \subseteq R$ an ideal and $t \geq 0$ a real number. We say that a pair $(R, \mathfrak{a}^t)$ is *(sharply) F-split* if there exists an $e > 0$ and an element

$$\phi \in (F^e_* \mathfrak{a}^{\lceil t(p^e - 1) \rceil}) \cdot \operatorname{Hom}_R(F^e_* R, R)$$

such that $\phi(F^e_* R) = R$. Prove the following facts about sharply F-split pairs.

i. Show that if $(R, \mathfrak{a}^t)$ is sharply F-split then for all sufficiently divisible $e > 0$, there exists a ϕ as above.
ii. Find an example that is sharply F-split but such that a ϕ as above does not exist for every $e > 0$.
iii. Suppose that $(R, \mathfrak{a}^t)$ is sharply F-split. Show that there exists $t' \geq t$ with $t' = \frac{b}{p^e-1}$ such that $(R, \mathfrak{a}^{t'})$ is also sharply F-split.
iv. Show that $(R, \mathfrak{a}^t)$ is sharply F-split if and only if for every maximal $\mathfrak{m} \in \operatorname{Spec} R$, $(R_\mathfrak{m}, \mathfrak{a}^t_\mathfrak{m})$ is sharply F-split.
v. If $(R, \mathfrak{m})$ is local, then show that $(R, \mathfrak{a}^t)$ is sharply F-split if and only if for some $e > 0$ there exists an element $z \in \mathfrak{a}^{\lceil t(p^e-1)\rceil}$ such that the map

$$\begin{aligned} R &\longrightarrow F^e_* R \\ 1 &\longmapsto F^e_* z \end{aligned}$$

splits as a map of R-modules.

(7) A map of rings $R \longrightarrow S$ is called *pure* if for every R-module M, the map $M \longrightarrow M \otimes_R S$ is injective. Suppose that S is a finite R-module, show that $R \longrightarrow S$ is pure if and only if $R \longrightarrow S$ splits as a map of R-modules.

Hint: First reduce to the case that R is complete and local and let E be the injective hull of the residue field. Consider the injective map $E \longrightarrow E \otimes_R S$ and apply Matlis duality and $\operatorname{Hom} - \otimes$ adjointness.

Remark. A ring is called *F-pure* if the Frobenius map $R \longrightarrow F_* R$ is pure. Because of this, and because most rings we study are F-finite, the terms F-pure or F-split are usually synonymous.

(8) Given an ideal $\mathfrak{a} \subseteq R$ in an F-finite ring, we define the *F-split (or F-pure) threshold* $\operatorname{fpt}(R, \mathfrak{a})$ *of* $(R, \mathfrak{a})$ to be

$$\operatorname{fpt}(R, \mathfrak{a}) := \{t \geq 0 \mid (R, \mathfrak{a}^t) \text{ is sharply } F\text{-split}\}.$$

Compute the F-pure threshold of the following pairs, in all of them $k = \overline{\mathbb{F}_p}$.

i. $k[x, y]$, $\mathfrak{a} = \langle x^u y^v \rangle$.
ii. $k[x, y, z]$, $\mathfrak{a} = \langle x^2 y - z^2 \rangle$ (the answer depends on the characteristic).
iii. $k[x, y]$, $\mathfrak{a} = \langle y^2 - x^3 \rangle$ (the answer also depends on the characteristic).