# EXERCISES FOR CHARACTERISTIC $p$ COMMUTATIVE ALGEBRA JANUARY 18TH, 2017

DUE, JANUARY 27TH, 2017

(1) Write down a basis for $F^e_* \mathbb{F}_p[x_1, \ldots, x_n]$ over $\mathbb{F}_p[x_1, \ldots, x_n]$.

(2) Write down a basis for $F^e_* \mathbb{F}_p(t)[x_1, \ldots, x_n]$ over $\mathbb{F}_p(t)[x_1, \ldots, x_n]$.

(3) If $I = \langle f_1, \ldots, f_t \rangle \subseteq R$ is an ideal in a ring of characteristic $p > 0$, we use $I^{[p^e]}$ to denote the ideal $\langle f_1^{p^e}, \ldots, f_t^{p^e} \rangle$. Show that $I^{[p^e]}$ is independent of the choice of generators $f_i$ and also show that the formulas:

$$IF^e_* R = F^e_* I^{[p^e]} \text{ and } IR^{1/p^e} = (I^{[p^e]})^{1/p^e}$$

hold.

(4) If $R = k[x^2, x^3]$, verify that $R^{1/p}$ is not a free $R$-module for any prime $p$.

(5) Use the fact that a Noetherian local ring is regular if and only if it has finite global dimension to prove that if $R$ is a regular local ring, then so is $R_Q$ for any $Q \in \operatorname{Spec} R$.

(6) Find the singular locus of the following rings $R$, try to simplify this ideal as much as possible (in particular, take the radical of the ideal defining the singular locus if you can).

(ie, find an ideal which defines the primes $Q$ in $\operatorname{Spec} R$ such that $R_Q$ is not a regular local ring. Note you'll have to figure out the dimension of each ring.).

i. $k = \overline{k}, R = k[x, y, z]/\langle y^3 - x^2 z \rangle$.

ii. $k = \overline{k}, R = k[x, y, z, w]/\langle z^2 - yw, yz - xw, y^2 - xz \rangle$.

iii. (you may want the help of a computer) $k = \overline{k}$,

$$R = k[x, y, z, w]/\langle yz - xw, z^3 - yw^2, xz^2 - y^2 w, y^3 - x^2 z \rangle.$$

(7) Consider the following singular local rings defined over $k = \overline{k}, \operatorname{char} k \neq 2$. See if you can identify which ones remain domains after completion at the identified maximal ideal.

i. $k[x, y]/\langle y^2 - x^3 \rangle$ localized at the origin $\langle x, y \rangle$.

ii. $k[x, y]/\langle y^2 - x^2(x - 1) \rangle$ localized at the origin $\langle x, y \rangle$.

iii. $k[x, y, z]/\langle xy^2 - z^2 \rangle$ localized at the origin $\langle x, y, z \rangle$.