

**MATH 7800 WORKSHEET #1**  
DUE JANUARY 16TH IN GRADESCOPE

1. Let  $R$  be a ring of prime characteristic  $p$  and  $W$  a multiplicative set in  $R$ . Suppose that  $M$  is any  $R$ -module. Then for every  $e \in \mathbb{N}$ , there is a natural  $W^{-1}R$ -module isomorphism

$$(1) \quad W^{-1}F_*^e M \longrightarrow F_*^e(W^{-1}M) \quad \frac{r}{w} \otimes F_*^e m \mapsto F_*^e \left( \frac{r^{p^e} m}{w^{p^e}} \right),$$

where the second  $F_*^e$  can be viewed as the Frobenius pushforward functor for either  $W^{-1}R$ -modules or for  $R$ -modules.

**2.** Let  $k$  be any field of characteristic  $p$ . Find a minimal generating set for  $F_*^e k[x_1, \dots, x_n]$  over  $k[x_1, \dots, x_n]$ , and prove that it is a *free basis*. If  $[k : k^p] = d < \infty$ , prove that the rank of both  $F_*^e k[x_1, \dots, x_n]$  over  $k[x_1, \dots, x_n]$  and  $F_*^e k[\![x_1, \dots, x_n]\!]$  over  $k[\![x_1, \dots, x_n]\!]$  agree and compute it.

3. Let  $k = \mathbb{F}_p(t_1, t_2, \dots)$  be a field generated over  $\mathbb{F}_p$  by infinitely many indeterminates  $t_i$ , and let  $R = k[x]$  by the polynomial ring over  $k$ .

- (1) Show that  $F_*R$  is not a finitely generated  $R$ -module.
- (2) Show that the canonical map (of rings)  $\widehat{R} \otimes F_*R \rightarrow \widehat{F_*R}$  is not an isomorphism, where the notation  $\widehat{M}$  denotes completion of the  $R$  module  $M$  at the maximal ideal  $(x)$ .

I asked ChatGPT (5.2, Utah's version) this question on January 5th, 2026. Its solution for (2) is below. Is it correct? If not, identify the error write a correct solution. For fun, try this problem on some other LLMs and report what you find.

Let  $\widehat{(\cdot)}$  denote completion at the maximal ideal  $(x) \subset R$ . For a Noetherian ring  $R$  and an  $R$ -module  $M$ , there is a canonical map

$$\widehat{R} \otimes_R M \longrightarrow \widehat{M}.$$

It is a standard fact that if  $M$  is finitely generated over  $R$ , then this map is an isomorphism. Moreover, since  $\widehat{R}$  is faithfully flat over  $R$ , the converse holds in the following sense: if the above map is an isomorphism and  $\widehat{R} \otimes_R M$  is finitely generated over  $\widehat{R}$ , then  $M$  must be finitely generated over  $R$ .

Now take  $M = F_*R$ . If the canonical map

$$\widehat{R} \otimes_R F_*R \longrightarrow \widehat{F_*R}$$

were an isomorphism, then  $\widehat{F_*R}$  would be a finitely generated  $\widehat{R}$ -module. By faithful flatness of  $\widehat{R}$  over  $R$ , this would imply that  $F_*R$  is a finitely generated  $R$ -module. This contradicts part (1), where we showed that  $F_*R$  is not finitely generated over  $R$ .

Therefore, the canonical map

$$\widehat{R} \otimes_R F_*R \longrightarrow \widehat{F_*R}$$

is not an isomorphism.

4. Suppose that  $A$  is a ring of characteristic  $p > 0$  and  $R$  is an  $A$ -algebra. The previous exercise demonstrated that  $F : R \rightarrow R$  is not generally a map of  $A$ -algebras. Define a map of  $F_*A$ -algebras

$$F_{R/A} : R \otimes_A (F_*A) \longrightarrow F_*R$$

$$(r \otimes F_*a) \longmapsto F_*(r^p a).$$

This map is called the *relative Frobenius of  $R$  over  $A$* . Show that it is a map of  $F_*A$ -algebras. Further show that for any  $A$ -algebra  $B$ , the base change map  $F_{R/A} \otimes_{F_*A} F_*B$  is equal to  $F_{R \otimes_A B/B}$ . In other words, the relative Frobenius is compatible with base change.

5. Consider  $A = \mathbb{F}_p[t]$  and  $R = A[x]/(x^p - t)$ . Write down explicitly the relative Frobenius map  $F_{R/A}$  and show that the source and target are not isomorphic as rings.

6. If  $(R, \mathfrak{m}) \xrightarrow{\phi} (S, \mathfrak{n})$  is a local homomorphism of Noetherian local rings, then  $\phi$  is flat if and only if the induced map on completions is  $\widehat{R}^{\mathfrak{m}} \rightarrow \widehat{S}^{\mathfrak{n}}$  if flat.

*Hint:* There are different ways to do this, but you might find either Tag 00HC or Tag 0AGW useful from the stacks project.

7. Suppose  $R$  is a perfect<sup>1</sup> (but not necessarily Noetherian) ring of characteristic  $p > 0$ . Let  $S = R[x_1, \dots, x_n]/I$  for some finitely generated ideal  $I$ . Show that  $F_*S$  is a finitely presented  $S$ -module.

I asked Utah's ChatGPT (version 5.2) this question on January 5th, 2026, and here is the response. Is it correct / complete?

*Proof.* Let  $R$  be a perfect ring of characteristic  $p > 0$ . Set

$$A := R[x_1, \dots, x_n], \quad S := A/I,$$

where  $I \subset A$  is a finitely generated ideal. Denote by  $F$  the Frobenius endomorphism. Write

$$A^{(1)} := F(A) = R[x_1^p, \dots, x_n^p] \subset A, \quad S^{(1)} := F(S) \subset S.$$

By definition,  $F_*S$  is the ring  $S$  viewed as an  $S$ -module via Frobenius; equivalently, the  $S$ -action on  $F_*S$  factors through  $S^{(1)}$ .

**Step 1:  $A$  is finite free over  $A^{(1)}$ .** For a multi-index  $\alpha = (\alpha_1, \dots, \alpha_n)$  with  $0 \leq \alpha_i \leq p-1$ , set

$$x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

Every monomial  $x_1^{m_1} \cdots x_n^{m_n}$  can be written uniquely as

$$(x_1^p)^{q_1} \cdots (x_n^p)^{q_n} x_1^{r_1} \cdots x_n^{r_n}, \quad 0 \leq r_i < p.$$

Hence every  $f \in A$  has a unique expression

$$f = \sum_{\alpha} g_{\alpha}(x_1^p, \dots, x_n^p) x^{\alpha}, \quad g_{\alpha} \in A^{(1)}.$$

Thus  $\{x^\alpha\}_{0 \leq \alpha_i < p}$  is an  $A^{(1)}$ -basis of  $A$ , so  $A$  is finite free of rank  $p^n$  over  $A^{(1)}$ .

**Step 2:  $S$  is finitely presented as an  $A^{(1)}$ -module.** Choose generators  $I = (f_1, \dots, f_m)$ . For each  $i$ , write

$$f_i = \sum_{\alpha} g_{i,\alpha}(x^p) x^{\alpha}, \quad g_{i,\alpha} \in A^{(1)},$$

with only finitely many nonzero coefficients. Consider the  $A^{(1)}$ -linear surjection

$$\Phi : (A^{(1)})^{\oplus p^n} \longrightarrow S, \quad e_{\alpha} \longmapsto \overline{x^{\alpha}}.$$

The kernel of  $\Phi$  is generated by the finitely many elements

$$\sum_{\alpha} g_{i,\alpha} e_{\alpha}, \quad i = 1, \dots, m,$$

corresponding to the relations  $f_i \in I$ . Hence  $S$  is finitely presented as an  $A^{(1)}$ -module.

**Step 3:  $S$  is finitely presented as an  $S^{(1)}$ -module.** The natural map  $A^{(1)} \rightarrow S^{(1)}$  is surjective. Tensoring the finite presentation of  $S$  over  $A^{(1)}$  with  $S^{(1)}$  yields

$$S \cong (S^{(1)})^{\oplus p^n} / (K \cdot S^{(1)})$$

for some finitely generated submodule  $K$ . Thus  $S$  is finitely presented as an  $S^{(1)}$ -module.

**Step 4: Conclusion.** Since the  $S$ -action on  $F_*S$  factors through  $S^{(1)}$ , the  $S$ -module  $F_*S$  corresponds to  $S$  viewed as an  $S^{(1)}$ -module. By Step 3 this module is finitely presented, and therefore  $F_*S$  is a finitely presented  $S$ -module.  $\square$

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<sup>1</sup>That is  $F : R \rightarrow R$  is a bijection.