

MATH 7800 WORKSHEET #1
DUE JANUARY 16TH IN GRADESCOPE

1. Let R be a ring of prime characteristic p and W a multiplicative set in R . Suppose that M is any R -module. Then for every $e \in \mathbb{N}$, there is a natural $W^{-1}R$ -module isomorphism

$$(1) \quad W^{-1}F_*^e M \longrightarrow F_*^e(W^{-1}M) \quad \frac{r}{w} \otimes F_*^e m \mapsto F_*^e \left(\frac{r^{p^e} m}{w^{p^e}} \right),$$

where the second F_*^e can be viewed as the Frobenius pushforward functor for either $W^{-1}R$ -modules or for R -modules.

2. Let k be any field of characteristic p . Find a minimal generating set for $F_*^e k[x_1, \dots, x_n]$ over $k[x_1, \dots, x_n]$, and prove that it is a *free basis*. If $[k : k^p] = d < \infty$, prove that the rank of both $F_*^e k[x_1, \dots, x_n]$ over $k[x_1, \dots, x_n]$ and $F_*^e k[[x_1, \dots, x_n]]$ over $k[[x_1, \dots, x_n]]$ agree and compute it.

3. Let $k = \mathbb{F}_p(t_1, t_2, \dots)$ be a field generated over \mathbb{F}_p by infinitely many indeterminates t_i , and let $R = k[x]$ by the polynomial ring over k .

(1) Show that F_*R is not a finitely generated R -module.

(2) Show that the canonical map (of rings) $\widehat{R} \otimes F_*R \rightarrow \widehat{F_*R}$ is not an isomorphism, where the notation \widehat{M} denotes completion of the R module M at the maximal ideal (x) .

I asked ChatGPT (5.2, Utah's version) this question on January 5th, 2026. Its solution for (2) is below. Is it correct? If not, identify the error write a correct solution. For fun, try this problem on some other LLMs and report what you find.

Let $\widehat{(\cdot)}$ denote completion at the maximal ideal $(x) \subset R$. For a Noetherian ring R and an R -module M , there is a canonical map

$$\widehat{R} \otimes_R M \longrightarrow \widehat{M}.$$

It is a standard fact that if M is finitely generated over R , then this map is an isomorphism. Moreover, since \widehat{R} is faithfully flat over R , the converse holds in the following sense: if the above map is an isomorphism and $\widehat{R} \otimes_R M$ is finitely generated over \widehat{R} , then M must be finitely generated over R .

Now take $M = F_*R$. If the canonical map

$$\widehat{R} \otimes_R F_*R \longrightarrow \widehat{F_*R}$$

were an isomorphism, then $\widehat{F_*R}$ would be a finitely generated \widehat{R} -module. By faithful flatness of \widehat{R} over R , this would imply that F_*R is a finitely generated R -module. This contradicts part (1), where we showed that F_*R is not finitely generated over R .

Therefore, the canonical map

$$\widehat{R} \otimes_R F_*R \longrightarrow \widehat{F_*R}$$

is not an isomorphism.

4. Suppose that A is a ring of characteristic $p > 0$ and R is an A -algebra. The previous exercise demonstrated that $F : R \rightarrow R$ is not generally a map of A -algebras. Define a map of F_*A -algebras

$$F_{R/A} : R \otimes_A (F_*A) \longrightarrow F_*R$$

$$(r \otimes F_*a) \longmapsto F_*(r^pa).$$

This map is called the *relative Frobenius of R over A* . Show that it is a map of F_*A -algebras. Further show that for any A -algebra B , the base change map $F_{R/A} \otimes_{F_*A} F_*B$ is equal to $F_{R \otimes_A B/B}$. In other words, the relative Frobenius is compatible with base change.

5. Consider $A = \mathbb{F}_p[t]$ and $R = A[x]/(x^p - t)$. Write down explicitly the relative Frobenius map $F_{R/A}$ and show that the source and target are not isomorphic as rings.

6. If $(R, \mathfrak{m}) \xrightarrow{\phi} (S, \mathfrak{n})$ is a local homomorphism of Noetherian local rings, then ϕ is flat if and only if the induced map on completions is $\widehat{R}^{\mathfrak{m}} \rightarrow \widehat{S}^{\mathfrak{n}}$ if flat.

Hint: There are different ways to do this, but you might find either Tag 00HC or Tag 0AGW useful from the stacks project.

7. Suppose R is a perfect¹ (but not necessarily Noetherian) ring of characteristic $p > 0$. Let $S = R[x_1, \dots, x_n]/I$ for some finitely generated ideal I . Show that F_*S is a finitely presented S -module.

I asked Utah's ChatGPT (version 5.2) this question on January 5th, 2026, and here is the response. Is it correct / complete?

Proof. Let R be a perfect ring of characteristic $p > 0$. Set

$$A := R[x_1, \dots, x_n], \quad S := A/I,$$

where $I \subset A$ is a finitely generated ideal. Denote by F the Frobenius endomorphism. Write

$$A^{(1)} := F(A) = R[x_1^p, \dots, x_n^p] \subset A, \quad S^{(1)} := F(S) \subset S.$$

By definition, F_*S is the ring S viewed as an S -module via Frobenius; equivalently, the S -action on F_*S factors through $S^{(1)}$.

Step 1: A is finite free over $A^{(1)}$. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ with $0 \leq \alpha_i \leq p-1$, set

$$x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

Every monomial $x_1^{m_1} \cdots x_n^{m_n}$ can be written uniquely as

$$(x_1^p)^{q_1} \cdots (x_n^p)^{q_n} x_1^{r_1} \cdots x_n^{r_n}, \quad 0 \leq r_i < p.$$

Hence every $f \in A$ has a unique expression

$$f = \sum_{\alpha} g_{\alpha}(x_1^p, \dots, x_n^p) x^{\alpha}, \quad g_{\alpha} \in A^{(1)}.$$

Thus $\{x^{\alpha}\}_{0 \leq \alpha_i < p}$ is an $A^{(1)}$ -basis of A , so A is finite free of rank p^n over $A^{(1)}$.

Step 2: S is finitely presented as an $A^{(1)}$ -module. Choose generators $I = (f_1, \dots, f_m)$. For each i , write

$$f_i = \sum_{\alpha} g_{i,\alpha}(x^p) x^{\alpha}, \quad g_{i,\alpha} \in A^{(1)},$$

with only finitely many nonzero coefficients. Consider the $A^{(1)}$ -linear surjection

$$\Phi : (A^{(1)})^{\oplus p^n} \longrightarrow S, \quad e_{\alpha} \longmapsto \overline{x^{\alpha}}.$$

The kernel of Φ is generated by the finitely many elements

$$\sum_{\alpha} g_{i,\alpha} e_{\alpha}, \quad i = 1, \dots, m,$$

corresponding to the relations $f_i \in I$. Hence S is finitely presented as an $A^{(1)}$ -module.

Step 3: S is finitely presented as an $S^{(1)}$ -module. The natural map $A^{(1)} \rightarrow S^{(1)}$ is surjective. Tensoring the finite presentation of S over $A^{(1)}$ with $S^{(1)}$ yields

$$S \cong (S^{(1)})^{\oplus p^n} / (K \cdot S^{(1)})$$

for some finitely generated submodule K . Thus S is finitely presented as an $S^{(1)}$ -module.

Step 4: Conclusion. Since the S -action on F_*S factors through $S^{(1)}$, the S -module F_*S corresponds to S viewed as an $S^{(1)}$ -module. By Step 3 this module is finitely presented, and therefore F_*S is a finitely presented S -module. \square

¹That is $F : R \rightarrow R$ is a bijection.