WS #4 – MATH 6320
SPRING 2015

DUE: TUESDAY FEBRUARY 24TH

You are encouraged to work in groups on this. Only one write up needs to be turned in per group. We learn about Hom in this worksheet.

1. Show that the functor $\mathrm{Hom}_R(\bullet, N)$ is left exact and contravariant. In other words, if

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is an exact sequence of R-modules, then

$$0 \longrightarrow \mathrm{Hom}_R(C, N) \xrightarrow{g'} \mathrm{Hom}_R(B, N) \xrightarrow{f'} \mathrm{Hom}_R(A, N)$$

is exact. (Did you use the fact that $A \longrightarrow B$ was injective?)

2. We prove a converse to **1.** Suppose that $A \longrightarrow B \longrightarrow C$ is a sequence of maps between R-modules and also that $0 \longrightarrow \mathrm{Hom}_R(C, N) \xrightarrow{g'} \mathrm{Hom}_R(B, N) \xrightarrow{f'} \mathrm{Hom}_R(A, N)$ is exact for *every* R-module N. Prove that $A \longrightarrow B \longrightarrow C \longrightarrow 0$ is exact.

Hint: The trick is clever choices of N to get different parts of the exactness.

3. Show that the functor $\mathrm{Hom}_R(M, \bullet)$ is right exact and covariant. In other words if

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is an exact sequence of R-modules, then show that

$$0 \longrightarrow \mathrm{Hom}_R(M, A) \xrightarrow{f''} \mathrm{Hom}_R(M, B) \xrightarrow{g''} \mathrm{Hom}_R(M, C)$$

is also exact. (Did you use the fact that $B \longrightarrow C$ was surjective?)

4. Now we prove a converse to **3**. Suppose that $A \longrightarrow B \longrightarrow C$ is a sequence of maps between R-modules and also that $0 \longrightarrow \mathrm{Hom}_R(M, A) \xrightarrow{f''} \mathrm{Hom}_R(M, B) \xrightarrow{g''} \mathrm{Hom}_R(M, C)$ is exact for *every* R-module M. Prove that $0 \longrightarrow A \longrightarrow B \longrightarrow C$ is exact.

Hint: Clever choices of M are the order of the day.

5. A short exact sequence $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ is called *split exact* if there is a map $\gamma : C \longrightarrow B$ such that $\beta \circ \gamma : C \longrightarrow B \longrightarrow C$ is the identity. Show that in that case $B \cong A \oplus C$ and α is identified with $a \mapsto (a, 0)$ and β is identified with $(a, c) \mapsto c$.

Fact: In this case, there is also a map $\delta : B \longrightarrow A$ so that $\delta \circ \alpha : A \longrightarrow B \longrightarrow A$ is the identity, which is also equivalent to showing that the short exact sequence is split exact.

A R-module P is called *projective* if for every *surjective* map of R-modules $f : A \longrightarrow B$ and every R-module map $g : P \longrightarrow B$, there exists a map $h : P \longrightarrow A$ such that the following diagram commutes:

$$\begin{array}{ccc} & & P \\ & {}^{h}\swarrow & \downarrow g \\ A & \xrightarrow[f]{} & B. \end{array}$$

6. Show that a free module $P = R^{\oplus(\ldots)}$ is projective. Further show that if P is projective and $P \cong A \oplus B$, then A is also projective.

7. Suppose that $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} P \longrightarrow 0$ is exact and P is projective, prove that

$$0 \longrightarrow \mathrm{Hom}_R(P, M) \longrightarrow \mathrm{Hom}_R(B, M) \longrightarrow \mathrm{Hom}_R(A, M) \longrightarrow 0$$

is exact for every R-module M.

Hint: Show that if P is projective, then the exact sequence is split exact.