WS #3 – MATH 6320
SPRING 2015

DUE: THURSDAY FEBRUARY 5TH

You are encouraged to work in groups on this. Only one writeup needs to be turned in per group. Let us again recall first what we know.

Theorem. *Let $G \subseteq \mathrm{Aut}(K)$ be a group with n elements acting on a field K. Let $F = K^G$ be the fixed field. Then $[K : F] = n = |G|$.*

Corollary. *Let K/F be a finite extension then $|\mathrm{Aut}(K/F)| \leq [K : F]$.*

Corollary. *Let $F = K^G$ as above. Then every element $\sigma \in \mathrm{Aut}(K)$ that fixes F is contained in G.*

Theorem. *Let K/F be a finite extension of fields. The following are equivalent:*

(1) K is the splitting field of a separable polynomial defined over F.
(2) $[K : F] = |\mathrm{Aut}(K/F)|$
(3) $F = K^G$ where $G = \mathrm{Aut}(K/F)$.
(4) K/F is Galois (you can take Galois to mean any of the above).

Our goal is to prove the rest of the following theorem (in particular parts (4) and (5)).

Theorem. (Fundamental theorem of Galois theory) *Let K/F be Galois and let $G = \mathrm{Gal}(K/F)$. Then there is a bijection between subfields $F \subseteq E \subseteq K$ and subgroups $\{1\} \leq H \leq G$. This is obtained by $H \mapsto K^H$ and $E \mapsto \{g \in G \mid \text{ which fix } E\ \}$. Furthermore:*

(1) This bijection is inclusion reversing ($H_1 \subseteq H_2$ if and only if $E_2 \subseteq E_1$).
(2) $[K : E] = |H|$ and $[E : F] = |G : H|$.
(3) K/E is always Galois with $\mathrm{Gal}(K/E) = H$.
(4) E is Galois over F if and only if H is a normal subgroup in G. In this case $\mathrm{Gal}(E/F) \cong G/H$.
(5) If E_1, E_2 correspond to H_1, H_2 then $E_1 \cap E_2$ corresponds to the group generated by H_1 and H_2. Likewise E_1E_2 corresponds to $H_1 \cap H_2$. In particular, the lattice of subfields is the same as the lattice of subgroups (upside down).

1. Using the notation of (4) (E corresponds to H), show that distinct embeddings of E into K (that fix F) are in bijection with cosets σH of H in G.

Hint: Given $\sigma, \sigma' \in G$, they give the same embedding of E into K if and only if $\sigma^{-1}\sigma'$ is the identity on E.

2. Let $\mathrm{Emb}(E/F)$ denote the embeddings of E into a (fixed) algebraic closure of F (equivalently into K since K/F is Galois) and which fix F. Show that $|\mathrm{Emb}(E/F)| = [G : H] = [E : F]$.

Hint: Use **1.**.

3. Show that E/F is Galois if and only if each embedding of E into K is an automorphism of E (ie, sends E back into E).

Hint: Note E/F is Galois if and only if $|\mathrm{Aut}(E/F)| = [E : F] = |\mathrm{Emb}(E/F)|$. Of course, each automorphism of E/F is obtained via some embedding... (why?)

4. Show that the supgroup of G fixing $\sigma(E)$ is $\sigma H \sigma^{-1}$

5. Prove E is Galois over F if and only if H is a normal subgroup in G (the first part of (4)).

6. Show that $G/H \cong \mathrm{Gal}(E/F)$ (which completes the proof of (4)).

Hint: We have already seen that cosets of H are identified with embeddings of E (which we view as automorphisms of E since these embeddings of E all send E back into E). Show that this gives it to you (ie, the group operations are ok).

Now we move onto (5).

7. Prove both parts of (5) (just write it down).