# HW #8 – MATH 6320
# SPRING 2015

DUE: TUESDAY APRIL 28TH

(1) Suppose $S = k[x_1, \ldots, x_r]$ and $f \in S$. If $\mathrm{in}_{\mathrm{lex}}(f) \in k[x_s, \ldots, x_r]$ for some $s$, then show that $f \in k[x_s, \ldots, x_r]$.

(2) Suppose $S = k[x_1, \ldots, x_r]$ and $f \in S$. If $f$ is homogeneous with $\mathrm{in}_{\mathrm{grevlex}}(f) \in \langle x_s, \ldots, x_r \rangle$, for some $s$, then show that $f \in \langle x_s, \ldots, x_r \rangle$.

(3) Say $S = k[x_1, \ldots, x_r]$ and that $F$ is a finitely generated free $S$-module with basis and a fixed monomial order $>$. Suppose that $g_1, \ldots, g_t \in M \subseteq F$. Suppose that $\mathrm{in}(g_1), \ldots, \mathrm{in}(g_t)$ generated $\mathrm{in}(M)$, then show that $g_1, \ldots, g_t$ generate $M$ and hence is a Gröbner basis for $M$.

(4) Suppose that $R = \mathbb{R}[x]$, $M$ and $N$ are finitely generated $R$-modules and $L = R/\langle x^2+1 \rangle \oplus R^2$. Suppose that $L \oplus M \cong L \oplus N$. Prove that $M \cong N$.

(5) Find an ideal $I$ in the ring $A = \mathbb{Z}[x]$ such that $A/I$ has exactly three prime ideals. Identify the ideals and justify your assertion.

(6) Let $G$ be a finite group of order $p^n$, where $p$ is prime and $n \geq 1$. Suppose $G$ acts on a finite set $S$. Let $S'$ be the subset of $S$ consisting of elements fixed by $G$:

$$S' = \{x \in S \mid gx = x \text{ for all } g \in G\}.$$

Prove that the order of $S'$ is congruent to the order of $S$ modulo $p$.

(7) Consider $f = x^6 + 3 \in \mathbb{Q}[x]$ and let $\alpha \in \mathbb{C}$ be a root of $f$. Set $E = \mathbb{Q}[\alpha]$.
  (a) Show that $E$ contains a primitive 6th root of unity.
  (b) Show that $E$ is Galois over $\mathbb{Q}$.
  (c) How many intermediate fields $\mathbb{Q} \subseteq F \subseteq E$ are there with $|F : \mathbb{Q}| = 3$