## HW #7 – MATH 6320 SPRING 2015

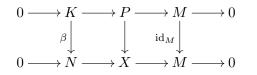
## DUE: THURSDAY APRIL 16TH

(10) Suppose that M, N are R-modules and that  $\operatorname{Ext}^1(M, N) = 0$ . Show that every extension  $0 \to N \to E \to M \to 0$  is split. More generally, identify a particular (obstruction) class in  $\operatorname{Ext}^1(M, N)$  which is zero if and only if a given extension  $0 \to N \to E \to M \to 0$  is split.

**Solution:** Given an extension  $0 \to N \to E \to M \to 0$ , apply  $\operatorname{Hom}_R(\bullet, N)$ . This gives us an exact sequence  $\operatorname{Hom}(E, N) \to \operatorname{Hom}(N, N) \to \operatorname{Ext}^1(M, N)$ . The image of the identity map  $\operatorname{id}_N : N \to N$ , is a particular element in  $\operatorname{Ext}^1(M, N)$ . If that image is zero, that means that there is a map  $\phi : E \to N$  such that  $\phi|_N = \operatorname{id}_N$ . In particular  $E = N \oplus M$  and our extension is trivial.

(Note, the original hint, while completely valid, made the next problem much harder.

- (11) We will show that  $\operatorname{Ext}^1(M, N)$  is in bijection with the set of equivalence classes of extensions  $0 \to N \to E \to M \to 0$ .
  - (a) Start with an exact sequence  $0 \to K \xrightarrow{i} P \xrightarrow{\rho} M \to 0$  with P projective and induce a map  $\partial$ : Hom $(K, N) \to \text{Ext}^1(M, N) \to 0$ . Thus each class x in Ext<sup>1</sup> gives us a (non-canonical)  $\beta \in \text{Hom}(K, N)$  with  $\partial(\beta) = x$ . Let X be the cokernel of  $K \to P \oplus N$ defined by  $k \mapsto (i(k), \beta(k))$ . Prove that there is a commutative diagram



with exact rows. Explain in particular what the map  $X \to M$  is.

**Solution:** Starting with  $0 \to K \xrightarrow{i} P \xrightarrow{\rho} M \to 0$  with P projective, obtain  $\partial$ : Hom $(K, N) \to \text{Ext}^1(M, N) \to 0$  as described. Let  $\beta$  be such that  $\partial(\beta) = x$ .

Let's show that  $\alpha : N \to X$  injects  $(n \mapsto \overline{(0,n)})$ . If  $n \in N$  satisfies  $\alpha(n) = 0$ , then  $(0,n) \in \ker(P \oplus N \to X)$ . Hence there is some  $k \in K$  with  $(i(k), \beta(k)) = (0,n)$ , but  $i: K \to P$  is injective, so n = 0 as claimed.

We define a map  $\kappa : X \to M$  as follows. For  $x \in X$ , choose (p, n) mapping to it. We define  $\kappa(x) = \rho(p)$ . We need to show it is well defined. Indeed, if (p, n) and (p', n') map to x, then there exists  $k \in K$  with  $(i(k), \beta(k)) = (p - p', n - n')$ . Hence  $p - p' = k \in K$ . In particular  $\rho(p) = \rho(p')$  and  $\kappa$  is well defined. Obviously  $\kappa$  is surjective. The image of  $N \to X$  is composed of (0, n). Such elements are mapped to by (0, n), which have image 0 in M obviously. Hence the kernel of  $\kappa$  contains the image of N.

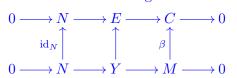
Of course, if  $(p,n) \in X$  maps to  $0 = \rho(p) \in M$ , then  $p = k \in K$ . Note that  $(p,n) = \overline{(p-k, n-\beta(k))} = \overline{(0, n-\beta(k))}$  and hence the image of N contains the kernel of K. This proves that the diagram above exists

(b) In the previous problem, you constructed an element in  $\text{Ext}^1(M, N)$  which was zero if and only if a given extension was split. Show that the element  $x \in \text{Ext}^1$  corresponds to the extension  $0 \to N \to X \to M \to 0$ .

**Solution:** Apply the functor  $Hom(\bullet, N)$  to the diagram in (a). This gives us a commutative diagram:

Now the diagram tells us that  $\partial \circ \delta = \mu$ , so since  $\partial(\beta) = x$  we want check that  $\delta(\mathrm{id}_N) = \beta$ . But  $\delta$  is induced from  $\beta$  so indeed we see that  $x = \partial(\beta) = \mu(\mathrm{id}_N)$ . Then we have shown that x corresponds to  $\mathrm{id}_N$  as claimed.

Alternately, if we had wanted to use the the identity map from the original hint in the previous problem  $(\operatorname{Hom}(M, E) \to \operatorname{Hom}(M, M) \to \operatorname{Ext}^1(M, N))$ , what we should have done is form  $0 \to N \to E \xrightarrow{\psi} C \to 0$  where E is an injective module. Then apply  $\operatorname{Hom}(M, \bullet)$  to that to get a surjection  $\operatorname{Hom}(M, C) \to \operatorname{Ext}^1(M, N) \to 0$ . A class  $x \in \operatorname{Ext}^1(M, N)$  gives us a map  $\beta \in \operatorname{Hom}(M, C)$ . One can then likewise make Y be the kernel of  $E \oplus M \xrightarrow{\psi - \beta} C$  and obtain a diagram:



Applying  $\text{Hom}(M, \bullet)$  and arguing as above gives us the extension mapping corresponding to x.

(c) Use what you have already done to complete the proof of the desired statement.

**Solution:** We are nearly done. Given an element of  $\operatorname{Ext}^1$ , we have shown how to produce an extension corresponding to it. This shows that the set map from extensions to elements of  $\operatorname{Ext}^1$  is surjective. To show it is injective, suppose that  $0 \to N \to X' \to M \to 0$  is an extension. Let  $0 \to K \to P \to M \to 0$  be another extension with P projective and note we have a map from one extension to the other just as in (b) (using the projectivity of P). A straightforward exercise can then show that X' is forced to be (isomorphic to the) cokernel of  $0 \to K \to P \oplus N$ . In particular, any two extensions corresponding to the same element of  $\operatorname{Ext}^1$  are equivalent.