HW #7 – MATH 6320
SPRING 2015

DUE: THURSDAY APRIL 16TH

(10) Suppose that M, N are R-modules and that $\mathrm{Ext}^1(M, N) = 0$. Show that every extension $0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0$ is split. More generally, identify a particular (obstruction) class in $\mathrm{Ext}^1(M, N)$ which is zero if and only if a given extension $0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0$ is split.

Solution: Given an extension $0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0$, apply $\mathrm{Hom}_R(\bullet, N)$. This gives us an exact sequence $\mathrm{Hom}(E, N) \longrightarrow \mathrm{Hom}(N, N) \longrightarrow \mathrm{Ext}^1(M, N)$. The image of the identity map $\mathrm{id}_N : N \longrightarrow N$, is a particular element in $\mathrm{Ext}^1(M, N)$. If that image is zero, that means that there is a map $\phi : E \longrightarrow N$ such that $\phi|_N = \mathrm{id}_N$. In particular $E = N \oplus M$ and our extension is trivial.

(Note, the original hint, while completely valid, made the next problem much harder.

(11) We will show that $\mathrm{Ext}^1(M, N)$ is in bijection with the set of equivalence classes of extensions $0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0$.

(a) Start with an exact sequence $0 \longrightarrow K \xrightarrow{i} P \xrightarrow{\rho} M \longrightarrow 0$ with P projective and induce a map $\partial : \mathrm{Hom}(K, N) \longrightarrow \mathrm{Ext}^1(M, N) \longrightarrow 0$. Thus each class x in Ext^1 gives us a (non-canonical) $\beta \in \mathrm{Hom}(K, N)$ with $\partial(\beta) = x$. Let X be the cokernel of $K \longrightarrow P \oplus N$ defined by $k \mapsto (i(k), \beta(k))$. Prove that there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & P & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \beta} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}_M} & & \\ 0 & \longrightarrow & N & \longrightarrow & X & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

with exact rows. Explain in particular what the map $X \longrightarrow M$ is.

Solution: Starting with $0 \longrightarrow K \xrightarrow{i} P \xrightarrow{\rho} M \longrightarrow 0$ with P projective, obtain $\partial : \mathrm{Hom}(K, N) \longrightarrow \mathrm{Ext}^1(M, N) \longrightarrow 0$ as described. Let β be such that $\partial(\beta) = x$.

Let's show that $\alpha : N \longrightarrow X$ injects ($n \mapsto \overline{(0, n)}$). If $n \in N$ satisfies $\alpha(n) = 0$, then $(0, n) \in \ker(P \oplus N \longrightarrow X)$. Hence there is some $k \in K$ with $(i(k), \beta(k)) = (0, n)$, but $i : K \longrightarrow P$ is injective, so $n = 0$ as claimed.

We define a map $\kappa : X \longrightarrow M$ as follows. For $x \in X$, choose (p, n) mapping to it. We define $\kappa(x) = \rho(p)$. We need to show it is well defined. Indeed, if (p, n) and (p', n') map to x, then there exists $k \in K$ with $(i(k), \beta(k)) = (p - p', n - n')$. Hence $p - p' = k \in K$. In particular $\rho(p) = \rho(p')$ and κ is well defined. Obviously κ is surjective. The image of $N \longrightarrow X$ is composed of $\overline{(0, n)}$. Such elements are mapped to by $(0, n)$, which have image 0 in M obviously. Hence the kernel of κ contains the image of N.
Of course, if $\overline{(p, n)} \in X$ maps to $0 = \rho(p) \in M$, then $p = k \in K$. Note that $\overline{(p, n)} = \overline{(p - k, n - \beta(k))} = \overline{(0, n - \beta(k))}$ and hence the image of N contains the kernel of K. This proves that the diagram above exists

(b) In the previous problem, you constructed an element in $\mathrm{Ext}^1(M,N)$ which was zero if and only if a given extension was split. Show that the element $x \in \mathrm{Ext}^1$ corresponds to the extension $0 \longrightarrow N \longrightarrow X \longrightarrow M \longrightarrow 0$.

Solution: Apply the functor $\mathrm{Hom}(\bullet, N)$ to the diagram in (a). This gives us a commutative diagram:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathrm{Hom}(M,N) & \longrightarrow & \mathrm{Hom}(P,N) & \longrightarrow & \mathrm{Hom}(K,N) & \xrightarrow{\partial} & \mathrm{Ext}^1(M,N) & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow{\scriptstyle \delta} & & \| & & \\ 0 & \longrightarrow & \mathrm{Hom}(M,N) & \longrightarrow & \mathrm{Hom}(X,N) & \longrightarrow & \mathrm{Hom}(N,N) & \xrightarrow[\mu]{} & \mathrm{Ext}^1(M,N) & \longrightarrow & \mathrm{Ext}^1(X,N) \end{array}$$

Now the diagram tells us that $\partial \circ \delta = \mu$, so since $\partial(\beta) = x$ we want check that $\delta(\mathrm{id}_N) = \beta$. But δ is induced from β so indeed we see that $x = \partial(\beta) = \mu(\mathrm{id}_N)$. Then we have shown that x corresponds to id_N as claimed.

Alternately, if we had wanted to use the the identity map from the original hint in the previous problem ($\mathrm{Hom}(M,E) \longrightarrow \mathrm{Hom}(M,M) \longrightarrow \mathrm{Ext}^1(M,N)$), what we should have done is form $0 \longrightarrow N \longrightarrow E \xrightarrow{\psi} C \longrightarrow 0$ where E is an injective module. Then apply $\mathrm{Hom}(M, \bullet)$ to that to get a surjection $\mathrm{Hom}(M,C) \longrightarrow \mathrm{Ext}^1(M,N) \longrightarrow 0$. A class $x \in \mathrm{Ext}^1(M,N)$ gives us a map $\beta \in \mathrm{Hom}(M,C)$. One can then likewise make Y be the kernel of $E \oplus M \xrightarrow{\psi - \beta} C$ and obtain a diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & C & \longrightarrow & 0 \\ & & {\scriptstyle \mathrm{id}_N}\uparrow & & \uparrow & & {\scriptstyle \beta}\uparrow & & \\ 0 & \longrightarrow & N & \longrightarrow & Y & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

Applying $\mathrm{Hom}(M, \bullet)$ and arguing as above gives us the extension mapping corresponding to x.

(c) Use what you have already done to complete the proof of the desired statement.

Solution: We are nearly done. Given an element of Ext^1, we have shown how to produce an extension corresponding to it. This shows that the set map from extensions to elements of Ext^1 is surjective. To show it is injective, suppose that $0 \longrightarrow N \longrightarrow X' \longrightarrow M \longrightarrow 0$ is an extension. Let $0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$ be another extension with P projective and note we have a map from one extension to the other just as in (b) (using the projectivity of P). A straightforward exercise can then show that X' is forced to be (isomorphic to the) cokernel of $0 \longrightarrow K \longrightarrow P \oplus N$. In particular, any two extensions corresponding to the same element of Ext^1 are equivalent.