HW #7 – MATH 6320
SPRING 2015

DUE: THURSDAY APRIL 16TH

(1) Suppose that M is a finitely generated module over a R. Suppose that $f : M \longrightarrow M$ is a surjective R-module homomorphism. Show that f is an isomorphism.

Hint: Nakayama's lemma is a good approach. View M as an $R[x]$ module where x acts via f.

(2) Suppose that M is a finitely generated projective module over a local ring $(R, \mathfrak{m})$. Show that M is actually a free module. Another way to say this is that *projective modules are locally free.*

Hint: Use the previous problem.

(3) Suppose that R is a ring and M is an R-module. Show that M is injective if and only if for every ideal $J \subseteq R$ and every R-module homomorphism $g : J \longrightarrow M$, g can be extended to $g' : R \longrightarrow M$.

Hint: Replacing $J \subseteq R$ with an arbitrary inclusion of modules gives you the definition of *injective*. Hence say $N' \subseteq N$ and $g : N' \longrightarrow M$ is a map. Consider the set of submodules of N to which g extends, use Zorn's lemma to make a biggest submodule of N to which g extends and then make a bigger one for a contradiction.

(4) Suppose that R is a ring.
  (a) If $W \subseteq R$ is a multiplicative set and I is an injective R-module, show that $W^{-1}I$ is an injective $W^{-1}R$-module.
  (b) Show that an arbitrary direct sum $\bigoplus_\lambda I_\lambda$ of injective R-modules is an injective R-module.

(5) Suppose that $\phi : R \longrightarrow S$ is a ring homomorphism and I is an injective R-module. Show that $\mathrm{Hom}_R(S, I)$ is an injective S-module.

Hint: This is not so hard, you need to prove/use a special case of a jazzed up version of Hom $-\otimes$ adjointness: for any S-module M, $\mathrm{Hom}_R(M, I) \cong \mathrm{Hom}_S(M, \mathrm{Hom}_R(S, I))$ as S-modules.

(6) If R is a PID. Show that an R-module I is injective if and only if it is divisible[1].

(7) Show that every $\mathbb{Z}$-module can be embedded into an injective $\mathbb{Z}$-module

Hint: First show that a free $\mathbb{Z}$-module can be embedded into an injective $\mathbb{Z}$-module. For M an arbitrary $\mathbb{Z}$-module, write $F \longrightarrow M \longrightarrow 0$ where F is a free $\mathbb{Z}$-module and $F \longrightarrow M$ has kernel K. Mod out some appropriate injective module by K.

(8) Conclude that for any ring R, any R-module can be embedded into an injective R-module. Hence show that injective resolutions exist. This result is an assertion that *the category of R-modules has enough injectives.*

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[1]For an integral domain R, an R-module M is called divisible if for every $0 \neq r \in R$, and every $m \in M$, there is some $m' \in M$ with $rm' = m$.

(10) Suppose that M, N are R-modules and that $\mathrm{Ext}^1(M,N) = 0$. Show that every extension $0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0$ is split. More generally, identify a particular (obstruction) class in $\mathrm{Ext}^1(M,N)$ which is zero if and only if a given extension $0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0$ is split.

Hint: Compute the long exact sequence associated to $\mathrm{Hom}(M, \bullet)$ and think about what it means. The class should be the image of particular identity map in Ext^1.

(11) We will show that $\mathrm{Ext}^1(M,N)$ is in bijection with the set of equivalence classes of extensions $0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0$.[2]

(a) Start with an exact sequence $0 \longrightarrow K \xrightarrow{i} P \xrightarrow{\rho} M \longrightarrow 0$ with P projective and induce a map $\partial : \mathrm{Hom}(K,N) \longrightarrow \mathrm{Ext}^1(M,N) \longrightarrow 0$. Thus each class x in Ext^1 gives us a (non-canonical) $\beta \in \mathrm{Hom}(K,N)$ with $\partial(\beta) = x$. Let X be the cokernel of $K \longrightarrow P \oplus N$ defined by $k \mapsto (i(k), \beta(k))$. Prove that there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] & P \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "\mathrm{id}_M"'] & 0 \\
0 \arrow[r] & N \arrow[r] & X \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Explain in particular what the map $X \longrightarrow M$ is.

(b) In the previous problem, you constructed an element in $\mathrm{Ext}^1(M,N)$ which was zero if and only if a given extension was split. Show that the element $x \in \mathrm{Ext}^1$ corresponds to the extension $0 \longrightarrow N \longrightarrow X \longrightarrow M \longrightarrow 0$.

(c) Use what you have already done to complete the proof of the desired statement.

[2] Remember, two extensions are equivalent if there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, "\mathrm{id}_N"'] & E \arrow[r] \arrow[d, "\sim"'] & M \arrow[r] \arrow[d, "\mathrm{id}_M"'] & 0 \\
0 \arrow[r] & N \arrow[r] & E' \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$