HW #5 – MATH 6320
SPRING 2015

DUE: TUESDAY MARCH 10TH

(1) Suppose that $W_1 \subseteq W_2$ are multiplicative sets in a ring R and M is an R-module. Show that $W_2^{-1}(W_1^{-1}M) \cong W_2^{-1}M$.

(2) Suppose that $N \subseteq M$ are R-modules and $W \subseteq R$ is a multiplicative set. Show directly that $W^{-1}(M/N) \cong W^{-1}M/W^{-1}N$. Here we are using the fact that $W^{-1}N$ can be identified with a submodule of $W^{-1}M$.

(3) We consider localizing at prime ideals.
   (a) Suppose that A is a ring and that for each prime ideal $P \in \mathrm{Spec}A$, the local ring $A_P := (A \setminus P)^{-1}A$ is an integral domain. Show that A need not be an integral domain.
   (b) However, if M and N are A-modules with a map $f : M \longrightarrow N$. Consider the induced map $f_P : (A \setminus P)^{-1}M =: M_P \longrightarrow N_P := (A \setminus P)^{-1}N$ for each $P \in \mathrm{Spec}A$. Show that f is surjective (respectively injective) if and only if f_P is surjective (respectively injective) for every $P \in \mathrm{Spec}A$.

(4) Suppose that R is an integral domain and $0 \neq r \in R$. Set $W = \{1, r, r^2, r^3, \ldots\}$. Is $R[r^{-1}] \cong W^{-1}R \cong R[x]/\langle xr - 1\rangle$ where x is an indeterminant?

(5) Suppose that k is a field. Consider the following set

$$S := \{f \in k[x] \mid f(0) = f(1)\}.$$

Show that S is a ring. Describe S as a quotient of a polynomial ring and describe its prime Spectrum. Additionally, describe the induced map on Specs from the inclusion of rings $S \hookrightarrow k[x]$ at least in the case that k is algebraically closed.

(6) Suppose that k is a field and that $f : R \longrightarrow S$ is a map between finitely generated k-algebras (this means that R is of the form $k[x_1, \ldots, x_n]/I$, and likewise with S, and also that f sends k to k). Show that the function $f^{\#} : \mathrm{Spec}S \longrightarrow \mathrm{Spec}R$ sends maximal ideals to maximal ideals.

Hint: You may use the fact that if K is a field and $L \supseteq K$ a field extension such that L is a finitely generated K-algebra, then L is a finite field extension. You may also use the fact that an integral domain which is a module finite extension of a field is itself a field.

(7) Suppose that A and B are rings. Figure out what $\mathrm{Spec}(A \oplus B)$ is in relation to $\mathrm{Spec}A$ and $\mathrm{Spec}B$.

(8) (*Warning: Hard*) Suppose that A and B are rings, I is an ideal of A with canonical projection $f : A \longrightarrow A/I$. Further suppose that we are given a ring homomorphism $g : B \longrightarrow A/I$. Consider the following set.

$$C = \{(a, b) \in A \oplus B | f(a) = g(b)\}.$$

   (a) Show that C is a subring of $A \oplus B$. Note that C has canoical maps to A and to B (call them p_1 and p_2 respectively).
   (b) Show that the elements of $\mathrm{Spec}C$ are in bijection with the set

$$\Big((\mathrm{Spec}A) \setminus V(I)\Big) \coprod (\mathrm{Spec}B).$$

Hint: Consider a prime in $\mathrm{Spec}C$. There are two possibilities, it either contains $\ker p_2$ or it does not. In the latter case, invert an appropriate element and analyze what happens.

(c) Describe geometrically Spec$C$ in the following examples where k is an algebraically closed field. Additionally describe the map Spec$A \longrightarrow$ Spec$C$ in each case:
   (i) $A := k[x]$, $I = \langle x^2 - 1\rangle$ and $B = k$.
   (ii) $A := k[x, y]$, $I = \langle x\rangle$ and $B = k$.

(d) Suppose now that g is surjective. Explain why Spec$C$ is the gluing of Spec$A$ to Spec$B$ along $V(I)$.

The following philosophical statement on the elements of C might help with this problem. C is made up of the functions in $A \oplus B$ that agree on the set $V(I)$.