HW #2 – MATH 6320
SPRING 2015

DUE: THURSDAY FEBRUARY 5TH

(1) Let $E = \mathbb{Q}(2^{1/2}, 2^{1/3})$. Find all subfields of E.

(2) Same setup as in the previous problem. Show that $E = \mathbb{Q}(2^{1/2} + 2^{1/3})$.

(3) Let $K = \mathbb{Q}\left(\frac{-1+\sqrt{-3}}{2}\right)$. Give an example of two non-isomorphic field extensions L_1 and L_2 of K such that $\mathrm{Gal}(L_1/K) \cong \mathrm{Gal}(L_2/K) \cong \mathbb{Z}/3\mathbb{Z}$. Justify your claims.

(4) Suppose that $f \in \mathbb{Q}[x]$ is an irreducible polynomial and that $\alpha, \beta \in \mathbb{C}$ are roots of f. Suppose that $\mathbb{Q} \subseteq K \subseteq \mathbb{C}$ is such that $K/\mathbb{Q}$ is a finite Galois extension. Show that $\mathbb{Q}[\alpha] \cap K$ is isomorphic to $\mathbb{Q}[\beta] \cap K$.

Hint: We know there is an isomorphism $\sigma : \mathbb{Q}[\alpha] \longrightarrow \mathbb{Q}[\beta]$ sending α to β. Show that σ extends to an automorphism of some larger field that sends K to K.

(5) Suppose that $\alpha \in \mathbb{C}$ with $\alpha^n \in \mathbb{Q}$ such that $\mathbb{Q}[\alpha]/\mathbb{Q}$ is Galois. Further suppose that F is the field containing $\mathbb{Q}$ generated by all the roots of unity in $\mathbb{Q}[\alpha]$. Show that $\mathrm{Gal}(\mathbb{Q}[\alpha] : F)$ is a cyclic group.

(6) Suppose that $K, L \subseteq \mathbb{C}$ are subfields, each being Galois over $\mathbb{Q}$. Show that the field KL, generated by K and L, is Galois over $\mathbb{Q}$.

(7) Same setup as in the previous problem. Now suppose that $[K : \mathbb{Q}]$ and $[L : \mathbb{Q}]$ are relatively prime. Compute $\mathrm{Gal}(KL/\mathbb{Q})$ in terms of $\mathrm{Gal}(K/\mathbb{Q})$ and $\mathrm{Gal}(L/\mathbb{Q})$. Deduce that there is a field E Galois over $\mathbb{Q}$ with $[E : \mathbb{Q}] = 44$.