WS #8 – MATH 6310
FALL 2019

Throughout we fix R to be a commutative ring and suppose M is an R-module.

Definitions. Suppose we take $P_\bullet \longrightarrow M$ ($\ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$) a projective resolution of M. Let $P_\bullet$ be the projective resolution itself (without the augmentation map to M), here we set $P_{-1} = 0$.

For another other module N, we define a functor $\mathrm{Ext}^i(\bullet, N)$:

$$\mathrm{Ext}^i(M,N) = \frac{\ker\left(\mathrm{Hom}(P_i,N) \longrightarrow \mathrm{Hom}(P_{i+1},N)\right)}{\mathrm{image}\left(\mathrm{Hom}(P_{i+1},N) \longrightarrow \mathrm{Hom}(P_{i-1},N)\right)}$$

and we define a functor $\mathrm{Tor}_i(\bullet, N)$:

$$\mathrm{Tor}_i(M,N) = \frac{\ker\left(P_i \otimes N \longrightarrow P_{i-1} \otimes N\right)}{\mathrm{image}\left(P_{i+1} \otimes N \longrightarrow P_i \otimes N\right)}$$

It can be shown that $\mathrm{Tor}_i(M,N)$ is isomorphic to $\mathrm{Tor}_i(N,M)$ (ie, we can take a projective resolution of either M or N).

Furthermore, if $N \longrightarrow I^\bullet$ ($0 \longrightarrow N \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \ldots$) is an injective resolution of N, with $I^\bullet$ the unaugmented resolution with $I^{-1} = 0$, we can alternately define a functor $\mathrm{Ext}^i(M, \bullet)$:

$$\mathrm{Ext}^i(M,N) = \frac{\ker\left(\mathrm{Hom}(M,I^i) \longrightarrow \mathrm{Hom}(M,I^{i+1})\right)}{\mathrm{image}\left(\mathrm{Hom}(M,I^{i-1}) \longrightarrow \mathrm{Hom}(M,I^i)\right)}$$

Ext defined this way is isomorphic to the Ext above.

Finally, if for any ideal $J \subseteq R$ (you normally need this ideal to be finitely generated) we define the functor $\Gamma_J(N) = \{x \in N \mid J^n x = 0 \text{ for } x \gg 0\}$. The the ith local cohomology of N with respect to J is defined to be the functor $H^i_J(\bullet)$:

$$H^i_J(N) = \frac{\ker\left(\Gamma_J(I^i) \longrightarrow \Gamma_J(I^{i+1})\right)}{\mathrm{image}\left(\Gamma_J(I^{i-1}) \longrightarrow \Gamma_J(I^i)\right)}.$$

All these definitions are independent of the projective or injective resolution chosen, by an argument similar to the one in the worksheet.

Facts. We have natural transformations of functors.

- $\mathrm{Ext}^0(M,N) \cong \mathrm{Hom}(M,N)$.
- $\mathrm{Tor}_0(M,N) \cong M \otimes N$.
- $H^0_J(N) \cong \Gamma_J(N)$.

We have the following vanishing results.

(a) If M is projective or N is injective, then $\mathrm{Ext}^i(M,N) = 0$ for all $i > 0$.
(b) If N is injective, then $H^i_J(N) = 0$ for all $i > 0$.
(c) If M or N is projective, then $\mathrm{Tor}_i(M,N) = 0$ for all $i > 0$.

If $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is a short exact sequence of R-modules, then we have the following long exact sequences (where most maps are just the ones induced by the functorial nature of Ext, Tor, etc.).

$$0 \to \mathrm{Hom}(C,N) \to \mathrm{Hom}(B,N) \to \mathrm{Hom}(A,N) \to \mathrm{Ext}^1(C,N) \to \mathrm{Ext}^1(B,N) \to \mathrm{Ext}^1(A,N) \to \mathrm{Ext}^2(C,N) \to \ldots \tag{1}$$

$$0 \to \mathrm{Hom}(M,A) \to \mathrm{Hom}(M,B) \to \mathrm{Hom}(M,C) \to \mathrm{Ext}^1(M,A) \to \mathrm{Ext}^1(M,B) \to \mathrm{Ext}^1(M,C) \to \mathrm{Ext}^2(M,A) \to \ldots \tag{2}$$

$$\cdots \to \mathrm{Tor}_2(C,N) \to \mathrm{Tor}_1(A,N) \to \mathrm{Tor}_1(B,N) \to \mathrm{Tor}_1(C,N) \to A \otimes N \to B \otimes N \to C \otimes N \to 0 \tag{3}$$

$$0 \to \Gamma_J(A) \to \Gamma_J(B) \to \Gamma_J(C) \to H^1_J(A) \to H^1_J(B) \to H^1_J(C) \to H^2_J(A) \to \ldots \tag{4}$$

We also saw that elements of $\mathrm{Ext}^1(M,N)$ correspond to extensions $0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0$ with the zero element of Ext^1 corresponding to the trivial = split extension.

1. Suppose R is an integral domain, that M is an R-module and $0 \neq x \in R$. Show that the map

$$M \xrightarrow{\cdot x} M$$

is injective (ie M is x-torsion-free) if and only if $\mathrm{Tor}_1(R/x, M) = 0$. In this case, x is called an M-regular element (sometimes people also assume that M-regular means also that the above move is not surjective).

2. Suppose R is an integral domain and $W \subseteq R$ is a multiplicative set. Prove that for any R-module M, that

$$\text{Tor}_i(W^{-1}R, M) = 0$$

for $i > 0$. Here we view $W^{-1}R$ as an R-module via the map $R \longrightarrow W^{-1}R$, $r \cdot \frac{a}{b} = \frac{ra}{b}$. *Hint:* First show it for $i = 1$ by taking a short exact sequence $0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$ where P is projective/free. You'll want to prove that if $0 \longrightarrow A \longrightarrow B$ is exact, then so is $0 \longrightarrow W^{-1}A \longrightarrow W^{-1}B$ is also exact, and then use problem **5.** from the previous homework (you may assume that isomorphism is actually a natural transformation of functors).

3. Fix some element $r \in R$. For any R-module M this induces a multiplication map $M \xrightarrow{\cdot r} M$. Prove that the induced map $\mathrm{Ext}^i(M,N) \xrightarrow{\mathrm{Ext}^i(\cdot r, N)} \mathrm{Ext}^i(M,N)$ can also be identified with multiplication by r. The analogous statement holds for the other functors above (you don't need to prove it).

4. Suppose that $\langle x_1, x_2\rangle \subseteq R$ is an ideal. Suppose that M is an R-module such that x_1 is not a zero divisor on M and x_2 is not a zero divisor on M/x_1M. Prove that $H^i_{\langle x_1,x_2\rangle}(M) = 0$ for $i = 0, 1$.

Hint: First show that $H^0_{\langle x_1,x_2\rangle}(M/x_1M) = 0$. Then show that any element of $H^1_{\langle x_1,x_2\rangle}(M)$ is annihilated by a high enough power of x_2.

A module is called *Cohen-Macaulay* at a maximal ideal $\mathfrak{m}$ if there elements $x_1, \ldots, x_n \subseteq \mathfrak{m}$ with $\mathfrak{m}^N \subseteq \langle x_1, \ldots, x_n\rangle$ (for some $N \gg 0$) and such that x_{i+1} is a non-zero divisor on $M/\langle x_1, \ldots, x_i\rangle M$ for each e. This is equivalent to various vanishing of Ext or local cohomology groups, as you might even imagine. For example, it is not difficult to see that if $R = \mathbb{Q}[x, y]$, then R is a Cohen-Macaulay module at $\langle x, y\rangle$.