WORKSHEET #6 – MATH 6310
FALL 2019

DUE: MONDAY, NOVEMBER 18TH

In this worksheet, we study classification of groups of small order.

Recall the following from worksheet #2.

Let N and H be groups and let $\phi : H \longrightarrow \mathrm{Aut}(N)$ be a group homomorphism. We get a left action of H on N by $h.n = (\phi(h))(n)$. Set

$$G = N \times H$$

with the following binary operation:

$$(n_1, h_1)(n_2, h_2) = (n_1(h_1.n_2), h_1 h_2).$$

This is called the *semi-direct product of N and H*, and is denoted by $N \rtimes H$.

Theorem. *Suppose G is a group with H an subgroup of G and N a normal subgroup of G. Further suppose that $NH = G$ and that $N \cap H = \{1\}$. Let $\phi : H \longrightarrow \mathrm{Aut}(N)$ be the map which conjugates by h. In other words $\phi(h)(n) = hnh^{-1}$. Then we have an isomorphism $N \rtimes H \cong G$.*

We can use this to classify groups as follows. Given some number n, suppose we can factor $n = ab$, with $\gcd(a, b) = 1$, where

- we can show (say by Sylow theorems) that groups G of order n have a normal subgroups N of order a.
- and groups G of order n have other subgroups H of order b.

Then any group of order n must be isomorphic to a semidirect product $N \rtimes H$. Thus to classify groups of order n, we need to classify such semidirect products.

1. Let $n = 10$. Show that any group of order 10 has a normal subgroup N of order 5 and a subgroup H of order 2. Classify all homomorphisms from $H \longrightarrow \mathrm{Aut}(N)$. How non-isomorphic groups of order 10 are there? (What happens when the homomorphism to $\mathrm{Aut}(N)$ is the trivial homomorphism?)

2. Let $n = 20$. Show that any group of order 20 has a normal subgroup N of order 5 and a subgroup H of order 4. Classify groups of order 4 (they are all Abelian), and use that to classify groups of order 20.

Hint: There can be two different homomorphisms $\phi : H \longrightarrow \mathrm{Aut}(N)$ that produce isomorphic semidirect products. You'll need to figure out why that happens, at least for this particular example.

3. Classify groups of order 12.

4. Show that if $n = p_1^{a_1} \dots p_l^{a_l}$ then $\mathrm{Aut}(\mathbb{Z}/n\mathbb{Z})$ is isomorphic to

$$U(\mathbb{Z}/n\mathbb{Z}) \cong U(\mathbb{Z}/p_1^{a_1}\mathbb{Z}) \oplus \cdots \oplus U(\mathbb{Z}/p_l^{a_l}\mathbb{Z})$$

where $U(R)$ is the group of units of a ring R. Additionally show that each $U(\mathbb{Z}/p_i^{a_i}\mathbb{Z})$ is cyclic if $p_i \neq 2$.

5. Classify groups of order 30 as follows. Show that every group of order 30 has a subgroup of order 15 which is necessarily normal. Next show/recall that groups of order 15 are cyclic. Next show that $\mathrm{Aut}(\mathbb{Z}/15) \cong \mathbb{Z}/2 \times \mathbb{Z}/4$ using the previous exercise. Find the elements of order 2 in that group of automorphisms. You can prove various groups you produce are nonisomorphic by looking at their centers.

6. Classify all groups of order 18.

7. Classify all groups of order 75.

8. Classify all groups of order 70.