WORKSHEET #4 – MATH 6310
FALL 2019

DUE: FRIDAY, OCTOBER 18TH

Definition: Suppose R is a ring. A *left R-module* M is an Abelian group under $+$ with a multiplication map by elements of R:

$$R \times M \longrightarrow M$$
$$(r, m) \longmapsto r.m$$

satisfying the following axioms.

(1) $r_1.(r_2.m) = (r_1 r_2).m$ for all $r_i \in R$, $m \in M$.
(2) $1.m = m$ for all $m \in M$.
(3) $r(m + n) = rm + rn$ for all $r \in R$, $m, n \in M$.
(4) $(r_1 + r_2)m = r_1 m + r_2 m$ for all $r_i \in R$, $m \in M$.

We can define a right R-module likewise (there is no real difference if R is commutative).

It is important to note that if R is a field, an R-module is nothing more than an R-vector space.

Another important example, the left ideals of R are exactly the R-submodules of R acting on itself.

1. Suppose that $\phi : R \longrightarrow S$ is a ring homomorphism and M is any S-module. Show that we can view M as an R-module via $r.m = \phi(r).m$.

2. Show that every Abelian group G under $+$ is a $\mathbb{Z}$-module.

3. Suppose R is a ring and M is an R-module. Define

$$\mathrm{Ann}_R(M) = \{r \in R \mid rm = 0 \; \forall m \in M\}.$$

Prove that this is an ideal of R and that we can also view M as an $R/\mathrm{Ann}_R(M)$-module.

4. Suppose that $N \subseteq M$ is a submodule of a module. Define the R-module M/N and prove the R-module action you define is well defined.

A homomorphism of (left) R-modules is a function $\phi : M \longrightarrow N$ between R-modules that is a homomorphism of Abelian groups (under addition) and such that $\phi(r.m) = r.\phi(m)$.

5. Formulate and prove the first isomorphism theorem for homomorphisms of left R-modules.

6. Suppose that $A \subseteq B \subseteq C$ are left R-modules. Prove that $C/B \cong (C/A)/(B/A)$ as left R-modules.

7. Suppose R is a commutative integral domain and that M is an R-module. Consider the set

$$M_{\text{tor}} = \{m \in M \mid rm = 0 \text{ for some nonzero } r \in R\}.$$

(a) Prove that M_{tor} is a submodule of M (called the *torsion submodule* of M).
(b) If $M_{\text{tor}} = \{0\}$ then M is called *torsion free*. Prove that M/M_{tor} is torsion free.

8. Suppose R is a commutative integral domain, $W \subseteq R$ is a multiplicative set and M is an R-module. Consider the set of pairs (m, w) with $m \in M$ and $w \in W$ and declare two pairs $(m, w) \sim (m', w')$ to be equivalent if there exists $v \in W$ such that $vw'm = vwm'$.

(a) Show that this defines an equivalence relation. The equivalence class of (m, w) is denoted by m/w.

(b) Prove that $W^{-1}M$ is a $W^{-1}R$-module (and also an R-module via the map $R \longrightarrow W^{-1}R$).

(c) If $R = \mathbb{Z}$, $W = \{1, 2, 2^2, 2^3, \dots\}$ and $M = \mathbb{Z}/6\mathbb{Z}$. Prove that $W^{-1}M$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$.