WORKSHEET #2 – MATH 6310
FALL 2019

DUE WEDNESDAY, SEPTEMBER 18TH, 2019

1. Suppose that G has a subgroup H of index n. Show that H contains a normal subgroup of G of index a divisor of $n!$.

2. Suppose G acts transitively in a finite set X and let H be a normal subgroup of G. Thus H acts on X too. Let $O_1, \ldots, O_r$ denote the distinct orbits of Hs action on X.

(a) Prove that G acts transitively on the set of H-orbits $\{O_i\}$ by left multiplication. Use this to deduce that all the orbits have the same cardinality.

(b) Prove that if $a \in O_1$ then $|O_1| = |H : H \cap \mathrm{Stab}_G(a)|$ and show that $r = |G : H \cdot \mathrm{Stab}_G(a)|$.

Let H and K be groups and let $\phi : K \longrightarrow \mathrm{Aut}(H)$ be a group homomorphism. We get a left action of K on H by $k.h = (\phi(k))(h)$. Set

$$G = H \times K$$

with the following binary operation:

$$(h_1, k_1)(h_2, k_2) = (h_1(k_1.h_2), k_1k_2).$$

This is called the *semi-direct product of* K *and* H, and is denoted by $H \rtimes K$.

3. Show that $G = H \rtimes K$ is a group.

4. If we identify H with $\{(h,1)\}$, then show that H is a *normal* subgroup of G. This helps explain the notation, the H is the normal factor in $H \rtimes K$.

5. Let $H = \mathbb{Z}/4\mathbb{Z}$ and $K = \mathbb{Z}/2\mathbb{Z}$. Let the map $\phi : K \longrightarrow \mathrm{Aut}(Hs)$ be the map which sends [1] to the inversion bijection. In other words, $\phi([1])(k) = -k$ and $\phi([0])(k) = k$. Show that $H \rtimes K$ is isomorphic D_4 (the dihedral group on the square).

6. Suppose G is a group with K an subgroup of G and H a normal subgroup of G. Further suppose that $HK = G$ and that $H \cap K = \{1\}$. Let $\phi : K \longrightarrow \mathrm{Aut}(H)$ be the map which conjugates by k. In other words $\phi(k)(h) = khk^{-1}$. Show that $G \cong H \rtimes K$.

Hint: First show that every element of HK can be written uniquely as hk for some $h \in H$ and $k \in K$. Then define a map $HK \longrightarrow H \rtimes K$ sending $hk \mapsto (h, k)$. You need to show that this is a homomorphism (it is obviously a bijection).