HW #6 – MATH 6310
FALL 2019

DUE: TUESDAY, NOVEMBER 26TH

1. Suppose R is a commutative ring. Consider the functor:

$$\bullet \mapsto \mathrm{Hom}_R(\mathrm{Hom}_R(\bullet, R), R).$$

This is called the *reflexification* functor.

Show that there is a natural transformation of the identity function to the reflexification functor. (Essentially, I am asking you to show that there is a map from any module M to the reflexification of M, and it is compatible with maps from $M \longrightarrow N$).

2. Suppose that $R \longrightarrow S$ is a map of commutative rings. Show that $\mathrm{Hom}_R(S, R)$ is an S-module. Then, in the case that $R = \mathbb{Z}$ and $S = \mathbb{Z}[i]$, prove that $\mathrm{Hom}_R(S, R)$ is a free S-module of rank 1.

3. Suppose R is a commutative ring. Given modules $B, \ldots, D, B', \ldots, D'$ and R-module maps $b, \ldots, d, a', \ldots, c', \beta, \ldots, \gamma$ as in the commutative diagram below:

$$\begin{array}{ccccccccc} & & B & \xrightarrow{b} & C & \xrightarrow{c} & D & \xrightarrow{d} & 0 \\ & & \big\downarrow{\scriptstyle \beta} & & \big\downarrow{\scriptstyle \gamma} & & \big\downarrow{\scriptstyle \delta} & & \\ 0 & \xrightarrow[a']{} & B' & \xrightarrow[b']{} & C' & \xrightarrow[c']{} & D' & & \end{array}$$

If the rows are exact, then we have an exact sequence

$$\ker\beta \longrightarrow \ker\gamma \longrightarrow \ker\delta \xrightarrow{\phi} \mathrm{coker}\beta \longrightarrow \mathrm{coker}\gamma \longrightarrow \mathrm{coker}\delta$$

where the maps between the kernels and cokernels are induced by b, c, b', c' and the map ϕ is magic (in the proof, the exactness that I really want to read about is the exactness at ϕ).

4. Suppose R is a commutative ring and $I, J \subseteq R$ are ideals. Prove that

$$(R/I) \otimes_R (R/J) \cong R/(I + J)$$

as R-modules.

5. Suppose that M is an R-module and $W \subseteq R$ is a multiplicative set. Prove that

$$W^{-1}M \cong (W^{-1}R) \otimes_R M$$

as R-modules. Here $W^{-1}M$ is as defined in

6. A short exact sequence of R-modules

$$0 \longrightarrow A \stackrel{\alpha}{\to} B \stackrel{\beta}{\to} C \longrightarrow 0$$

is called *split* if either of the following equivalent conditions hold.

(a) There exists $f : B \longrightarrow A$ such that $A \stackrel{\alpha}{\to} B \stackrel{f}{\to} A$ is an isomorphism.

(b) There exists $g : C \longrightarrow B$ such that $C \stackrel{g}{\to} B \stackrel{\beta}{\to} C$ is an isomorphism.

Prove they are indeed equivalent. Then show if the short exact sequence is split, that $B \cong \alpha(A) \oplus C$.

7. Suppose that $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is split. For any R-module M, prove that

$$0 \longrightarrow \mathrm{Hom}(M, A) \longrightarrow \mathrm{Hom}(M, B) \longrightarrow \mathrm{Hom}(M, C) \longrightarrow 0$$
$$0 \longrightarrow \mathrm{Hom}(C, M) \longrightarrow \mathrm{Hom}(B, M) \longrightarrow \mathrm{Hom}(A, M) \longrightarrow 0$$
$$0 \longrightarrow M \otimes A \longrightarrow M \otimes B \longrightarrow M \otimes C \longrightarrow 0$$

are all exact sequences.

8. Suppose that $A \longrightarrow B \longrightarrow C$ is a sequence of maps between R-modules and also that $0 \longrightarrow \mathrm{Hom}_R(C, N) \xrightarrow{g'} \mathrm{Hom}_R(B, N) \xrightarrow{f'} \mathrm{Hom}_R(A, N)$ is exact for *every* R-module N. Prove that $A \longrightarrow B \longrightarrow C \longrightarrow 0$ is exact.

Hint: The trick is clever choices of N to get different parts of the exactness.

The following is also true: Suppose that $A \longrightarrow B \longrightarrow C$ is a sequence of maps between R-modules and also that $0 \longrightarrow \mathrm{Hom}_R(M, A) \xrightarrow{f''} \mathrm{Hom}_R(M, B) \xrightarrow{g''} \mathrm{Hom}_R(M, C)$ is exact for *every* R-module M. Then $0 \longrightarrow A \longrightarrow B \longrightarrow C$ is exact. I won't ask you to prove it however.