HW #5 – MATH 6310
FALL 2019

DUE: WEDNESDAY, NOVEMBER 13TH

1. Suppose $D = k[x]$ where k is a field and M is a finitely generated D-module. Prove that
$$M \cong D/\langle f_1(x)\rangle \oplus \cdots \oplus D/\langle f_l(x)\rangle$$
where $f_i | f_{i-1}$ for $i = 2, \ldots, l$ and where the f_i are monic polynomials. Furthermore, show that this decomposition is unique.

In many texts, rational canonical form also means that associated polynomials of the block diagonal divide each other in the way above (or maybe $f_i | f_{i+1}$).

2. Determine the possible characteristic and minimal polynomials of an $n \times n$ matrix over $\mathbb{C}$ that has rank 1.

3. Let G be a finite Abelian group under addition and let $n > 0$ be an integer. Show that the map $\phi : G \longrightarrow G$ which sends $x \mapsto nx$ is a homomorphism, further show it is an isomorphism if and only if $\gcd(|G|, n) = 1$.

4. Determine, up to conjugacy, all 3×3 matrices M over $\mathbb{Q}$ that satisfy $M^3 = 2M^2$.

5. Suppose that k is the field with 3 elements and let $R = k[x]$. Identify eleven different (up to isomorphism) R-modules M such that $|M| = 9$.

6. Determine up to similarity, all *real* 5×5-matrices with characteristic polynomial $x(x^2+1)^2$.

7. Let M be a 3×3 complex matrix with $M^6 = M^4$ and $M^4 + M^2 = 2M^3$. Determine the possible Jordan forms of M.

8. Write down the rational and Jordan canonical forms of the following matrix (viewed over $\mathbb{C}$)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & -2 & 0 & 1 \\ -2 & 0 & -1 & -2 \end{bmatrix}.$$

Hint: Let $V = \mathbb{C}^4$ and view $\mathbb{C}[x]$ acting on V where x acts as that matrix. The 4 $\mathbb{C}$-basis elements generate V, but there are relations between them when viewed over $\mathbb{C}[x]$.

9. Compute the characteristic polynomial, the minimal polynomial, the Jordan normal form, and the rational canonical form of the matrix.

$$\begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}.$$