HW #4 – MATH 6310
FALL 2019

DUE: FRIDAY, OCTOBER 25TH

1. Show that the set $\mathbb{Z}[\sqrt{2}] \subseteq \mathbb{R}$ of real numbers of the form $m + n\sqrt{2}$ with $m, n \in \mathbb{Z}$, is a Euclidean domain with respect to the function $\delta(m + n\sqrt{2}) = |m^2 - 2n^2|$.

2. Show that if D is a PID and $0 \neq a \in D$, then $D/(a)$ is a field if a is irreducible and $D/(a)$ is not an integral domain if a is not irreducible.

3. Prove the following irreducibility criterion of Eisenstein. If $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{Z}[x]$ and there exists a prime $p \in \mathbb{Z}$ such that $p | a_i$ for all $i < n$ and $p^2 \not| a_0$, then $f(x)$ is irreducible in $\mathbb{Q}[x]$.

Hint: Factor $f = g \cdot h$. Consider what the leading terms and constant terms of g and h. Reduce them modulo p as we did in the proof of Gauss' Lemma.

4. Suppose D is a commutative integral domain which is not a field. Prove that $D[x]$ is not a PID.

5. Suppose that R is a commutative ring and M is an R-module. Suppose that $\phi : M \longrightarrow R^{\oplus n}$ is surjective. Show that there exists a map $\psi : R^{\oplus n} \longrightarrow M$ such that $\phi(\psi(x)) = x$.

Show further that M is isomorphic to $R^{\oplus n} \oplus N$ for some other R-module N.

Hint: Let N be the kernel of ϕ. Write down a map from $N \oplus R^{\oplus n}$ to M and show it is an isomorphism.

6. Suppose that p is prime. Show that $\mathbb{Z}/\langle p^e \rangle$ is not a direct sum of any two nonzero submodules. Is the same true for $\mathbb{Z}$?

7. Determine $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/\langle m \rangle, \mathbb{Z}/\langle n \rangle)$ for all $m, n > 0$.

8. A left R-module M is called *irreducible* if $M \neq 0$ and 0 and M are the only submodules of M. Show that $0 \neq 0M$ is irreducible if M is cyclic with ever non-zero element a generator.

9. A left ideal $I \subseteq R$ is called *maximal* if $R \neq I$ and there is no left ideal I' with $I \subsetneq I' \subsetneq R$. Show that M is irreducible if and only if $M \cong R/I$ where I is a maximal left ideal of R.

10. Show that if M_1 and M_2 are irreducible left R-modules, then any nonzero homomorphism $M_1 \longrightarrow M_2$ is an isomorphism. Conclude that if M is irreducible, then $\mathrm{End}_R(M) = \mathrm{Hom}_R(M, M)$ is a division ring.