HW #3 – MATH 6310
FALL 2019

DUE: FRIDAY, OCTOBER 4TH

Definition 1. Suppose R is a commutative integral domain. A subset $W \subseteq R$ containing 1 and not containing 0 is called a *multiplicative set* if for any $a, b \in W$, we have $ab \in W$. (Note the conditions on 1 and 0 are not always assumed in the literature, we make them here for convenience).

In this case, we define $W^{-1}R$ to be the set of pairs (a, b) such that $a \in R$ and $b \in W$, modulo the equivalence relation where $(a_1, b_1) \sim (a_2, b_2)$ if $a_1 b_2 = a_2 b_1$. The equivalence class of (a, b) we denote by a/b.

$W^{-1}R$ forms a ring under the familiar rules for adding and multiplying fractions (you may take this as given).

Furthermore, the map

$$\phi : R \longrightarrow W^{-1}R$$

sending $r \mapsto r/1$ is a ring homomorphism (you may also take it as given).

(1) Suppose that R is a commutative integral domain, $W \subseteq R$ is a multiplicative set and $Q \subseteq R$ is a prime ideal such that $W \cap Q = \emptyset$. Show that the set

$$W^{-1}Q := \{x/w \mid x \in Q, w \in W\} \subseteq W^{-1}R$$

is a prime ideal of $W^{-1}R$.

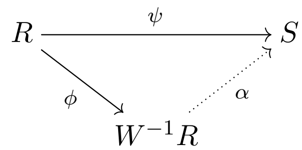
Hint: We checked this in class.

(2) Suppose that R is a commutative integral domain and $W \subseteq R$ is a multiplicative set. Suppose that $P \subseteq W^{-1}R$ is a prime ideal and set $Q = P \cap R$ (where the intersection is taken with the isomorphic copy of $R \subseteq W^{-1}(R)$). Show that $Q \cap W = \emptyset$ and that $W^{-1}Q = P$.

Conclude that the prime ideals of $W^{-1}R$ are in bijection with the prime ideals Q in R such that $W \cap Q = \emptyset$.

(3) Suppose that $W \subseteq R$ is a multiplicative set in a commutative integral domain, and W is made up of only units (= invertible elements) of R. Show that the map $\phi : R \longrightarrow W^{-1}R$ from the first page is an isomorphism of rings.

(4) Suppose that R is a commutative integral domain, $W \subseteq R$ is a multiplicative set and $\psi : R \longrightarrow S$ is a ring homomorphism between commutative rings. Further suppose that for each $w \in W$, there is some $s \in S$ such that $s\psi(w) = 1_S$. Show that one can factor ψ as follows



and that the map α is unique. This is called the *universal property of localization.*

(5) Suppose that R is a commutative integral domain and pick a nonzero $r \in R$. Set $W = \{1, r, r^2, r^3, \dots\}$. Show that we have an isomorphism between rings $R[x]/\langle rx - 1\rangle \cong W^{-1}R$.

(6) Show that any finite integral domain is a division ring.

(7) Do either of these distributive laws, for ideals I, J, K in a commutative ring R hold?

$$I \cdot (J + K) = I \cdot J + I \cdot K$$
$$I \cap (J + K) = I \cap J + I \cap K$$

(8) Suppose R is a commutative ring and $I, J \subseteq R$ are ideals such that $I + J = R$ (in this case we say the ideals are (strongly) relatively prime). Prove that $I \cap J = I \cdot J$. What does this say for ideals in $\mathbb{Z}$?

(9) Suppose that R is a ring with ideals $I_1, I_2, \ldots, I_k$ ideals such that $I_i + I_j = R$ for $i \neq j$. Show that the system of equations

$$\begin{aligned} x + I_1 &= a_1 + I_1 \\ x + I_2 &= a_2 + I_2 \\ \ldots &= \ldots \\ x + I_k &= a_k + I_k \end{aligned}$$

always has a solution.

Hint: First prove the case when $k = 2$ and show that two solutions differ by an element of $I_1 \cap I_2$. You can then do induction and at some point you'll also want to show that $I_j + \bigcap_{i \neq j} I_i = R$.

(10) Suppose that $I_1, I_2, \ldots, I_k \subseteq R$ are ideals such that $I_i + I_j = R$ for $i \neq j$. Prove that we have an isomorphism:

$$R/(I_1 \cap I_2 \cap \cdots \cap I_k) \cong (R/I_1) \oplus (R/I_2) \oplus \cdots \oplus (R/I_k)$$