HW #2 – MATH 6310
FALL 2019

DUE: FRIDAY, SEPTEMBER 13TH

(1) Recall that the *index* of a subgroup $H \leq G$ is $[G : H]$, the number of cosets (left = right) of H. Show that every subgroup of index 2 is normal. Conclude that A_n is normal in S_n.

(2) Suppose that H and H' are simple groups[1] and set $G = H \times H'$ a group with the product (componentwise) group structure. Show that every normal subgroup of G that is not $\{1\}$ or G,is isomorphic to H or H'.

Hint: Show that the intersection of normal subgroups is normal. Note, that not every subgroup will be of the form $H \times \{1\}$ and $\{1\} \times H'$ (in particular, think of the case where $H = \mathbb{Z}/p\mathbb{Z} = H'$ under addition).

[1]Groups with no proper nontrivial normal subgroups.

(3) Compute the cosets of $H = \langle (12) \rangle$ in S_3. Show explicitly with an example that addition of cosets via the formula $(aH)(bH) = (ab)H$ is *not* well defined.

(4) Suppose $S \subseteq G$ is a subset of a group such that $gSg^{-1} \subseteq S$ for all $g \in G$. Show that $\langle S \rangle$ is normal. Next let $T \subseteq G$ be another subset and form the set $V = \bigcup_{g \in G} g^{-1}Tg$. Show that $\langle V \rangle$ is the unique smallest normal subgroup of G containing T.

(5) Determine AutS_3. Here Aut(G) is the *group* of bijective group homomorphisms $\phi : G \longrightarrow G$ (isomorphisms) under composition.

(6) For any $a \in G$ consider the map $\phi_a : G \longrightarrow G$ defined by $\phi_a(x) = axa^{-1}$.
  (a) Show that ϕ_a is an automorphism. (It is called an *inner automorphism*).
  (b) Show that $a \mapsto \phi_a$ gives us a homomorphism $G \longrightarrow \mathrm{Aut}(G)$ with kernel equal to $Z(G)$ the center[2] of G.
  (c) Let $\mathrm{Inn}(G)$ be the image of the map in (ii). Show that $\mathrm{Inn}(G) \cong G/Z(G)$.
  (d) Finally show that $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.

[2]Those elements of G that commute with every other element of G

(7) **Extra credit:** Consider the free group F on the set $\{x_2, \ldots, x_n\}$. Consider the normal subgroup K generated by the following elements $x_i^2, (x_i x_j)^3, (x_i x_j x_i x_k)^2$ (for i, j, k distinct). Show that F/K is isomorphic to S_n.