WS #4 – MATH 6310
FALL 2017

DUE: MONDAY, DECEMBER 4TH

You are encouraged to work in groups on this. Only one write up needs to be turned in per group. We learn about Hom in this worksheet. Today, *ALL* rings are commutative.

1. Show that the functor $\mathrm{Hom}_R(\bullet, N)$ is left exact and contravariant. In other words, if

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is an exact sequence of R-modules, then

$$0 \longrightarrow \mathrm{Hom}_R(C, N) \xrightarrow{g'} \mathrm{Hom}_R(B, N) \xrightarrow{f'} \mathrm{Hom}_R(A, N)$$

is exact. (Did you use the fact that $A \longrightarrow B$ was injective?)

2. We prove a converse to **1.** Suppose that $A \longrightarrow B \longrightarrow C$ is a sequence of maps between R-modules and also that $0 \longrightarrow \mathrm{Hom}_R(C, N) \xrightarrow{g'} \mathrm{Hom}_R(B, N) \xrightarrow{f'} \mathrm{Hom}_R(A, N)$ is exact for *every* R-module N. Prove that $A \longrightarrow B \longrightarrow C \longrightarrow 0$ is exact.

Hint: The trick is clever choices of N to get different parts of the exactness.

3. Show that the functor $\mathrm{Hom}_R(M, \bullet)$ is right exact and covariant. In other words if

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is an exact sequence of R-modules, then show that

$$0 \longrightarrow \mathrm{Hom}_R(M, A) \xrightarrow{f''} \mathrm{Hom}_R(M, B) \xrightarrow{g''} \mathrm{Hom}_R(M, C)$$

is also exact. (Did you use the fact that $B \longrightarrow C$ was surjective?)

4. Now we prove a converse to **3**. Suppose that $A \longrightarrow B \longrightarrow C$ is a sequence of maps between R-modules and also that $0 \longrightarrow \mathrm{Hom}_R(M, A) \xrightarrow{f''} \mathrm{Hom}_R(M, B) \xrightarrow{g''} \mathrm{Hom}_R(M, C)$ is exact for *every* R-module M. Prove that $0 \longrightarrow A \longrightarrow B \longrightarrow C$ is exact.

Hint: Clever choices of M are the order of the day.

5. A short exact sequence $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ is called *split exact* if there is a map $\gamma : C \longrightarrow B$ such that $\beta \circ \gamma : C \longrightarrow B \longrightarrow C$ is the identity. Show that in that case $B \cong A \oplus C$ and α is identified with $a \mapsto (a, 0)$ and β is identified with $(a, c) \mapsto c$.

Fact: In this case, there is also a map $\delta : B \longrightarrow A$ so that $\delta \circ \alpha : A \longrightarrow B \longrightarrow A$ is the identity, which is also equivalent to showing that the short exact sequence is split exact.

A R-module P is called *projective* if for every *surjective* map of R-modules $f : A \longrightarrow B$ and every R-module map $g : P \longrightarrow B$, there exists a map $h : P \longrightarrow A$ such that the following diagram commutes:

$$\begin{array}{ccc} & & P \\ & \overset{h}{\swarrow} & \downarrow g \\ A & \xrightarrow[f]{} & B. \end{array}$$

6. Show that a free module $P = R^{\oplus(\dots)}$ is projective. Further show that if P is projective and $P \cong A \oplus B$, then A is also projective.

7. Suppose that $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} P \longrightarrow 0$ is exact and P is projective, prove that

$$0 \longrightarrow \mathrm{Hom}_R(P, M) \longrightarrow \mathrm{Hom}_R(B, M) \longrightarrow \mathrm{Hom}_R(A, M) \longrightarrow 0$$

is exact for every R-module M.

Hint: Show that if P is projective, then the exact sequence is split exact.