WORKSHEET #2 – MATH 6310
FALL 2017

DUE WEDNESDAY OCTOBER 18TH

We'll have some fun with localization.

Definition 1. Suppose R is a integral domain. A subset $W \subseteq R$ containing 1 and not containing 0 is called a *multiplicative set* if for any $a, b \in W$, we have $ab \in W$. (Note the conditions on 1 and 0 are not always assumed in the literature, we make them here for convenience).

1. Suppose that $Q \subseteq R$ is a prime ideal. Show that $R \setminus Q$ is a multiplicative set. In particular, note that $R \setminus \{0\}$ is a multiplicative set in an integral domain.

Definition 2. Suppose $W \subseteq R$ is a multiplicative set in an integral domain. Consider the set of all pairs $\{(r, w) \mid r \in R, w \in W\}$ under the following equivalence relation $(r, w) \sim (r', w')$ if $w'r = wr'$. The set of equivalence classes is denoted $W^{-1}R$ and equivalence classes of pairs (r, w) are denoted by r/w.

2. Suppose that $W \subseteq R$ is a multiplicative set. Show that $W^{-1}R$ is a ring under the operations induced by the formulae

$$r/w + r'/w' = (r'w + rw')/(ww') \text{ and } (r/w)(r'/w') = (rr'/ww').$$

In particular, show that those operations are well defined. Also show that the map $\phi : R \longrightarrow W^{-1}R$ defined by $\phi(r) = r/1$ is an injective ring homomorphism.

3. If R is an integral domain and $W = R \setminus \{0\}$ show that $W^{-1}R$ is a field. This field is called the field of fractions of R.

4. If R is an integral domain, $W = R \setminus \{0\}$, and $K = W^{-1}R$ is the field of fractions, show that K contains an isomorphic copy of R as a subring.

5. Now suppose that R is an integral domain, $W \subseteq R$ is a multiplicative set and $Q \subseteq R$ is a prime ideal such that $W \cap Q = \emptyset$. Show that the set

$$W^{-1}Q := \{x/w \mid x \in Q, w \in W\} \subseteq W^{-1}R$$

is a prime ideal of $W^{-1}R$.

6. Suppose that R is an integral domain and $W \subseteq R$ is a multiplicative set. Suppose that $P \subseteq W^{-1}R$ is a prime ideal and set $Q = P \cap R$ (where the intersection is taken with the isomorphic copy of $R \cong \phi(R)$ from **2.**). Show that $Q \cap W = \emptyset$ and that $W^{-1}Q = P$.

7. Conclude that the prime ideals of $W^{-1}R$ are in bijection with the prime ideals Q in R such that $W \cap Q = \emptyset$.

8. Suppose that $W \subseteq R$ is a multiplicative set in an integral domain made up of only units (= invertible elements) of R. Show that the map $\phi : R \longrightarrow W^{-1}R$ you constructed in **2.** is an isomorphism.

9. Give an example of an integral domain and two multiplicative sets $W, V \subseteq R$ such that $W \neq V$, $W^{-1}R \cong V^{-1}R$ but $\phi : R \longrightarrow W^{-1}R$ is not surjective.

10. Suppose that R is an integral domain, $W \subseteq R$ is a multiplicative set and $\psi : R \longrightarrow S$ is a ring homomorphism between commutative rings. Further suppose that for each $w \in W$, there is some $s \in S$ such that $s\psi(w) = 1_S$. Show that one can factor ψ as follows

$$\begin{tikzcd} R \arrow[rr, "\psi"] \arrow[dr, "\phi"'] & & S \\ & W^{-1}R \arrow[ur, dotted, "\alpha"'] & \end{tikzcd}$$

and that the map α is unique. This is called the *universal property of localization.*