NOTES – MATH 6320
FALL 2017

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1. Friday, December 1st, 2017

As before, all rings are commutative.

1.1. Tensor products, continued.

Proposition 1.1. *If* $0 \longrightarrow L \xrightarrow{a} M \xrightarrow{b} N \longrightarrow 0$ *is an exact sequence and* T *is another* R*-module, then*

$$L \otimes_R T \xrightarrow{\alpha} M \otimes_R T \xrightarrow{\beta} N \otimes_R T \longrightarrow 0$$

is also exact.

This proposition asserts that $\otimes$ is *right-exact* (it takes short exact sequences to sequences that are exact on the right).

Proof. It is easy to see that β is surjective, indeed if $n \otimes t \in N \otimes T$, then since $M \longrightarrow N$ is surjective, there exists $m \in M$ such that $b(m) = n$. Hence $m \otimes t \mapsto n \otimes t$ and it follows that β surjects.

We now need to show that $\ker \beta = \operatorname{im} \alpha$. Let $C = \operatorname{im} \alpha$, we already know that $C \subseteq \ker \beta$ and so we have a map $\gamma : (M \otimes_R T)/C \longrightarrow N \otimes_R T$. It is sufficient to show that this map is injective. Define a map

$$\sigma : N \otimes_R T \longrightarrow (M \otimes_R T)/C$$

by $n \otimes t \mapsto \overline{b_1(n) \otimes t}$ where $b_1(n)$ is *any* $m \in M$ with $b(m) = n$ and $\overline{\bullet}$ denotes the image after modding out by C. We need to show that σ is well defined. Suppose that m and m' are such that $b(m) = b(m') = n$ then we need to show that $\overline{m \otimes t} = \overline{m' \otimes t}$ (this is the same as showing that the obvious bi-linear map from the universal property is well defined). But since $b(m) = b(m')$, there exists $l \in L$ such that $a(l) = m - m'$. Therefore since $a(l) \otimes t \in C$, we see that $\overline{(m - m') \otimes t} = 0$ and $\overline{m \otimes t} = \overline{m' \otimes t}$. This shows σ is well defined. Now, $(M \otimes_R T)/C \xrightarrow{\gamma} N \otimes_R T \xrightarrow{\sigma} (M \otimes_R T)/C$ sends $\overline{m \otimes t}$ back to itself. It follows that γ is injective. ∎

The functors of Hom and tensor are closely related.

Theorem 1.2 (Hom $-\otimes$ adjointness)**.** *If* L, M, N *are* R*-modules, then there is an* R*-module isomorphism:*

$$\operatorname{Hom}_R(L \otimes_R M, N) \cong \operatorname{Hom}_R(L, \operatorname{Hom}_R(M, N))$$

Proof. Given $\phi \in \mathrm{Hom}_R(L, \mathrm{Hom}_R(M, N))$ we need to construct $\Phi(\phi) \in \mathrm{Hom}_R(L, \mathrm{Hom}_R(M, N))$. We an action of ϕ on elements of $L \otimes_R M$. Given $\sum l_i \otimes m_i$ we define

$$\phi(\sum l_i \otimes m_i) = \sum (\phi(l_i))(m_i)$$

Note that each $\phi(l_i) \in \mathrm{Hom}_R(M, N)$ so it makes perfect sense to act upon m_i. Thus we have defined Φ.

To go the other way, suppose that $\psi \in \mathrm{Hom}_R(L \otimes_R M, N)$, and we will define $\Psi(\psi) \in \mathrm{Hom}_R(L, \mathrm{Hom}_R(M, N))$. So choose $l \in L$. Then $\Psi(\psi)(l) = \psi(l \otimes \_\_)$ where the blank is to be filled in from M.

We should verify that $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are the identities. But I will leave this to you (I think we'll do one direction as a class). $\square$

Now we discuss a proof of the right exactness of $\otimes$ via the left exactness of Hom.

Lemma 1.3. *$\otimes_R M$ is right exact for any R-module M.*

Proof. We suppose that $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is exact and we want to show that

$$A \otimes_R M \longrightarrow B \otimes_R M \longrightarrow C \otimes_R M \longrightarrow 0$$

is exact. It is sufficient to show that

$$0 \longrightarrow \mathrm{Hom}_R(C \otimes_R M, N) \longrightarrow \mathrm{Hom}_R(B \otimes_R M, N) \longrightarrow \mathrm{Hom}_R(A \otimes_R M, N)$$

is exact for any R-module N (by the worksheet). But that is exact if and only if

$$0 \longrightarrow \mathrm{Hom}_R(M, \mathrm{Hom}_R(C, N)) \longrightarrow \mathrm{Hom}_R(M, \mathrm{Hom}_R(B, N)) \longrightarrow \mathrm{Hom}_R(M, \mathrm{Hom}_R(A, N))$$

is exact by the adjointness of tensor and Hom (we also need to know that the adjointness isomorphism is compatible with morphisms in the M variable, but it is, I won't check it though). But this is exact if

$$0 \longrightarrow \mathrm{Hom}_R(C, N) \longrightarrow \mathrm{Hom}_R(B, N) \longrightarrow \mathrm{Hom}_R(A, N)$$

is exact, which follows if $A \longrightarrow B \longrightarrow C \longrightarrow 0$ is exact (which it is). $\square$

The rest of the class we spent on the worksheet.