NOTES – MATH 6320
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As before, all rings are commutative. Before getting to Tor and Ext, we need a lemma.

Lemma 1.1 (Snake Lemma)**.** *Given modules $B, \dots, D, B', \dots, D'$ and R-module maps*

$$b, \dots, d, a', \dots, c', \beta, \dots, \gamma$$

as in the commutative diagram below:

$$\begin{array}{ccccccc} & & B & \xrightarrow{b} & C & \xrightarrow{c} & D & \xrightarrow{d} & 0 \\ & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & & \\ 0 & \xrightarrow[a']{} & B' & \xrightarrow[b']{} & C' & \xrightarrow[c']{} & D' & & \end{array}$$

If the rows are exact, then we have an exact sequence

$$\ker\beta \longrightarrow \ker\gamma \longrightarrow \ker\delta \xrightarrow{\phi} \operatorname{coker}\beta \longrightarrow \operatorname{coker}\gamma \longrightarrow \operatorname{coker}\delta$$

where the maps between the kernels and cokernels are induced by b, c, b', c' and the map ϕ is magic (the exactness at ϕ is the key point).

Proof. I'll sketch the start of it in class. The rest is an exercise, or see the movie "It's my turn". □

2. Tor and Ext

We have so far seen several functors which are left or right exact, but not *exact*. For instance

- $\operatorname{Hom}_R(M, \_\_)$ is left exact
- $\operatorname{Hom}_R(\_\_, N)$ is left exact (although contravariant)
- $\otimes$ is right exact

It turns out there is a nice way to handle all these failures of exactness and many others. Through the use of derived functors.

First a formality.

Definition 2.1. Suppose that $B^\bullet = \ldots \to B^{-1} \to B^0 \to B^1 \to B^2 \to \ldots$ is a complex (ie, $\ker(B_i \longrightarrow B_{i+1}) \supseteq \operatorname{im}(B_{i-1} \longrightarrow B_i)$). We define the ith cohomology of $B^\bullet$ to be

$$\mathbf{h}^i(B^\bullet) = \ker(B_i \longrightarrow B_{i+1}) / \operatorname{im}(B_{i-1} \longrightarrow B_i)$$

2.1. Tor.

Definition 2.2 (Projective resolutions)**.** Suppose R is a ring and M is an R-module. A projective resolution of M is a series of projective (ie free) modules P_i, $i = 0, -1, -2, \dots$ and maps

$$\ldots \xrightarrow{f_n} P^{-n} \xrightarrow{f_{n-1}} P^{-n+1} \xrightarrow{f_{n-2}} \ldots \xrightarrow{f_2} P^{-2} \xrightarrow{f_1} P^{-1} \xrightarrow{f_0} P^0 \to M \longrightarrow 0$$

making the above sequence exact. Such a sequence could be infinite. Since every module is a quotient of a free (and hence projective) module, every module has a projective resolution (although not a unique one).

Definition 2.3 (Tor)**.** Suppose that $P^\bullet \longrightarrow M$ is a projective resolution of M. Note that for any module C, $P^\bullet \otimes C$ is a complex. We define $\mathrm{Tor}_i(M, C)$ to be $\mathbf{h}^i(P^\bullet \otimes C)$. It is not obvious that this is independent of the choice of projective resolution, but it is true.

It is easy to see that:

Lemma 2.4. $\mathrm{Tor}_0(M, C) \cong M \otimes C$. *Furthermore, if M is projective then* $\mathrm{Tor}_i(M, C) = 0$ *for all* $i > 0$.

One other fact that is useful, but which we won't prove is that

Lemma 2.5. $\mathrm{Tor}_i(M, C) \cong \mathrm{Tor}_i(C, M)$.

Now suppose that $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ is a short exact sequence. We form a projective resolutions of L and N to form the following:

$$
\begin{array}{ccccccccc}
 & & \dots & & & & \dots & & \\
 & & \downarrow & & & & \downarrow & & \\
 & & P'^{-2} & & & & P''^{-2} & & \\
 & & \downarrow & & & & \downarrow & & \\
 & & P'^{-1} & & & & P''^{-1} & & \\
 & & \downarrow & & & & \downarrow & & \\
 & & P'^{0} & & & & P''^{0} & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0
\end{array}
$$

We set $P_i = P'_i \oplus P''_i$ with the canonical short exact sequences $0 \longrightarrow P'_i \longrightarrow P_i \longrightarrow P''_i \longrightarrow 0$. We claim that these combine to form a commutative diagram

$$
\begin{array}{ccccccccc}
 & & \dots & & \dots & & \dots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P'^{-2} & \longrightarrow & P^{-2} & \longrightarrow & P''^{-2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P'^{-1} & \longrightarrow & P^{-1} & \longrightarrow & P''^{-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P'^{0} & \longrightarrow & P^{0} & \longrightarrow & P''^{0} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0
\end{array}
$$

where the columns form projective resolutions. This is pretty easy but care must be taken with the details here.

Now apply the functor $\otimes_R C$ for some module C to the resolutions $P^\bullet = \dots P^2 \to P^1 \to P^0$ (likewise with $P'^\bullet$ and $P''^\bullet$). We obtain

$$\begin{array}{ccccccccc}
 & & \dots & & \dots & & \dots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P'^{-2} \otimes C & \longrightarrow & P^{-2} \otimes C & \longrightarrow & P''^{-2} \otimes C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P'^{-1} \otimes C & \longrightarrow & P^{-1} \otimes C & \longrightarrow & P''^{-1} \otimes C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P'^{0} \otimes C & \longrightarrow & P^{0} \otimes C & \longrightarrow & P''^{0} \otimes C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

We now apply the snake lemma, first to the diagram

$$\begin{array}{ccccccccc}
 & & P'^{-1} \otimes C/\operatorname{im}(P'^{-2} \otimes C) & \longrightarrow & P^{-1} \otimes C/\operatorname{im}(P^{-2} \otimes C) & \longrightarrow & P''^{-1} \otimes C/\operatorname{im}(P''^{-2} \otimes C) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P'^{0} \otimes C & \longrightarrow & P^{0} \otimes C & \longrightarrow & P''^{0} \otimes C & \longrightarrow & 0
\end{array}$$

The cokernel below the bottom row is simply

$$L \otimes C \longrightarrow M \otimes C \longrightarrow N \otimes C \longrightarrow 0$$

but this snakes up and connects with the kernels above the top row, which are

$$\operatorname{Tor}_1(L, C) \longrightarrow \operatorname{Tor}_1(M, C) \longrightarrow \operatorname{Tor}_1(N, C)$$

connecting these we get a long exact sequence

$$\operatorname{Tor}_1(L, C) \longrightarrow \operatorname{Tor}_1(M, C) \longrightarrow \operatorname{Tor}_1(N, C) \longrightarrow L \otimes C \longrightarrow M \otimes C \longrightarrow N \otimes C \longrightarrow 0.$$

But we don't stop now. We next consider the diagram:

$$\begin{array}{ccccccccc}
 & & P'^{-2} \otimes C/\operatorname{im}(P'^{-3} \otimes C) & \longrightarrow & P^{-2} \otimes C/\operatorname{im}(P^{-3} \otimes C) & \longrightarrow & P''^{-2} \otimes C/\operatorname{im}(P''^{-3} \otimes C) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker(P'^{-1} \otimes C \longrightarrow P'^{0} \otimes C) & \longrightarrow & \ker(P^{-1} \otimes C \longrightarrow P^{0} \otimes C) & \longrightarrow & \ker(P''^{-1} \otimes C \longrightarrow P''^{0} \otimes C) & &
\end{array}$$

applying the snake lemma again (and again) gets us to the long exact sequence

$$\begin{array}{ccccccc}
 & & & & \dots & \longrightarrow & \\
\operatorname{Tor}_2(L, C) & \longrightarrow & \operatorname{Tor}_2(M, C) & \longrightarrow & \operatorname{Tor}_2(N, C) & \longrightarrow & \\
\operatorname{Tor}_1(L, C) & \longrightarrow & \operatorname{Tor}_1(M, C) & \longrightarrow & \operatorname{Tor}_1(N, C) & \longrightarrow & \\
L \otimes C & \longrightarrow & M \otimes C & \longrightarrow & N \otimes C & \longrightarrow & 0.
\end{array}$$

2.2. **Ext.**

We first consider the functor $\operatorname{Hom}_R(\_\_, C)$.

Definition 2.6. If $P^\bullet$ is a projective resolution of M, then we define $\operatorname{Ext}^i(M, C)$ to be $\mathbf{h}^i(\operatorname{Hom}_R(P^\bullet, C))$, the ith cohomology of the complex $\operatorname{Hom}_R(P^\bullet, C)$.

Given a short exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

as above, we again form projective resolutions

$$\begin{array}{ccccccccc}
 & & \dots & & \dots & & \dots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P'^{-2} & \longrightarrow & P^{-2} & \longrightarrow & P''^{-2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P'^{-1} & \longrightarrow & P^{-1} & \longrightarrow & P''^{-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P'^{0} & \longrightarrow & P^{0} & \longrightarrow & P''^{0} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0
\end{array}$$

and apply the functor $\mathrm{Hom}_R(\_\_, C)$ to the projective resolutions to obtain:

$$\begin{array}{ccccccccc}
 & & \dots & & \dots & & \dots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longleftarrow & \mathrm{Hom}_R(P'^{-2}, C) & \longleftarrow & \mathrm{Hom}_R(P^{-2}, C) & \longleftarrow & \mathrm{Hom}_R(P''^{-2}, C) & \longleftarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longleftarrow & \mathrm{Hom}_R(P'^{-1}, C) & \longleftarrow & \mathrm{Hom}_R(P^{-1}, C) & \longleftarrow & \mathrm{Hom}_R(P''^{-1}, C) & \longleftarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longleftarrow & \mathrm{Hom}_R(P'^{0}, C) & \longleftarrow & \mathrm{Hom}_R(P^{0}, C) & \longleftarrow & \mathrm{Hom}_R(P''^{0}, C) & \longleftarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0
\end{array}$$

Applying the same snake lemma formalisms again, we note that we have diagrams

$$\begin{array}{ccccccc}
\ker\begin{pmatrix} \mathrm{Hom}_R(P'^{-i-1}, C) \\ \longrightarrow \\ \mathrm{Hom}_R(P'^{-i-2}, C) \end{pmatrix} & \longleftarrow & \ker\begin{pmatrix} \mathrm{Hom}_R(P^{-i-1}, C) \\ \longrightarrow \\ \mathrm{Hom}_R(P^{-i-2}, C) \end{pmatrix} & \longleftarrow & \ker\begin{pmatrix} \mathrm{Hom}_R(P''^{-i-1}, C) \\ \longrightarrow \\ \mathrm{Hom}_R(P''^{-i-2}, C) \end{pmatrix} & \longleftarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \\
0 \leftarrow \mathrm{Hom}_R(P'^{-i}, C)/\operatorname{im}(\mathrm{Hom}_R(P'^{-i+1}, C)) & \leftarrow & \mathrm{Hom}_R(P^{-i}, C)/\operatorname{im}(\mathrm{Hom}_R(P^{-i+1}, C)) & \leftarrow & \mathrm{Hom}_R(P''^{-i}, C)/\operatorname{im}(\mathrm{Hom}_R(P''^{-i+1}, C)) & &
\end{array}$$

The snake lemma yields the following long exact sequence.

$$\begin{array}{rccccc}
0 \longrightarrow & \mathrm{Hom}_R(N, C) & \longrightarrow & \mathrm{Hom}_R(M, C) & \longrightarrow & \mathrm{Hom}_R(L, C) \\
\longrightarrow & \mathrm{Ext}^1_R(N, C) & \longrightarrow & \mathrm{Ext}^1_R(M, C) & \longrightarrow & \mathrm{Ext}^1_R(L, C) \\
\longrightarrow & \mathrm{Ext}^2_R(N, C) & \longrightarrow & \dots & &
\end{array}$$

However, there isn't just one Ext functor... We also have $\mathrm{Hom}_R(B, \_\_)$. Projective resolutions just aren't good enough any more. We need

Definition 2.7 (Injective resolutions)**.** Suppose that M is a module. We say that

$$\left(0 \longrightarrow M \longrightarrow I^{\bullet}\right) = \left(0 \longrightarrow M \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow\right)$$

is an injective resolution if each I^i is an injective module and the above sequence is a long exact sequence. It is a non-trivial fact that injective resolutions exist (to show it, it is enough to show that for every module N, there is an injective module and an injection $N \hookrightarrow I$).

Definition 2.8. Fix $I^\bullet$ to be an injective resolution of a module M and let $\mathrm{Hom}_R(B, I^\bullet)$ be the corresponding complex. Then we define $\mathrm{Ext}^i_R(B, M)$ to be $\mathbf{h}^i(\mathrm{Hom}_R(B, I^\bullet))$.

There are a couple key facts we won't prove.

- This Ext is also independent of the choice of injective resolution.
- This Ext agrees with the other Ext we defined (which is really useful!) In other words
$$\mathbf{h}^i(\mathrm{Hom}_R(B, I^\bullet)) \cong \mathbf{h}^i(\mathrm{Hom}_R(P^\bullet, M))$$
where $I^\bullet$ is an injective resolution of M and $P^\bullet$ is a projective resolution of B.

Again, given $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ we can form injective resolutions of L and N and take the direct sum to get an injective resolution of M and so have

$$\begin{array}{ccccccccc}
 & & \cdots & & \cdots & & \cdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & I'^2 & \longrightarrow & I^2 & \longrightarrow & I''^2 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & I'^1 & \longrightarrow & I^1 & \longrightarrow & I''^1 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & I'^0 & \longrightarrow & I^0 & \longrightarrow & I''^0 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0
\end{array}$$

We can apply the covariant functor $\mathrm{Hom}_R(B, \_\_)$ to the I parts and obtain:

$$\begin{array}{ccccccccc}
 & & \cdots & & \cdots & & \cdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathrm{Hom}_R(B, I'^2) & \longrightarrow & \mathrm{Hom}_R(B, I^2) & \longrightarrow & \mathrm{Hom}_R(B, I''^2) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathrm{Hom}_R(B, I'^1) & \longrightarrow & \mathrm{Hom}_R(B, I^1) & \longrightarrow & \mathrm{Hom}_R(B, I''^1) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathrm{Hom}_R(B, I'^0) & \longrightarrow & \mathrm{Hom}_R(B, I^0) & \longrightarrow & \mathrm{Hom}_R(B, I''^0) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

The rows are exact and the columns are complexes. Using the snake lemma as before gives us a long exact sequence

$$\begin{array}{ccccccc}
0 & \to & \mathrm{Hom}_R(B, L) & \to & \mathrm{Hom}_R(B, M) & \to & \mathrm{Hom}_R(B, N) \\
 & \to & \mathrm{Ext}^1_R(B, L) & \to & \mathrm{Ext}^1_R(B, M) & \to & \mathrm{Ext}^1_R(B, N) \\
 & \to & \mathrm{Ext}^2_R(B, L) & \to & \mathrm{Ext}^2_R(B, M) & \to & \mathrm{Ext}^2_R(B, N) \\
 & \to & \cdots & & & &
\end{array}$$