NOTES – MATH 6320 FALL 2017

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1. Monday, December 4th, 2017

As before, all rings are commutative. Before getting to Tor and Ext, we need a lemma.

Lemma 1.1 (Snake Lemma). Given modules $B, \ldots, D, B', \ldots, D'$ and R-module maps

$$b, \ldots, d, a', \ldots, c', \beta, \ldots, \gamma$$

as in the commutative diagram below:

$$B \xrightarrow{b} C \xrightarrow{c} D \xrightarrow{d} 0$$

$$\downarrow^{\beta} \qquad \downarrow^{\gamma} \qquad \downarrow^{\delta}$$

$$0 \xrightarrow{a'} B' \xrightarrow{b'} C' \xrightarrow{c'} D'$$

If the rows are exact, then we have an exact sequence

$$\ker \beta \longrightarrow \ker \gamma \longrightarrow \ker \delta \xrightarrow{\phi} \operatorname{coker} \beta \longrightarrow \operatorname{coker} \gamma \longrightarrow \operatorname{coker} \delta$$

where the maps between the kernels and cokernels are induced by b, c, b', c' and the map ϕ is magic (the exactness at ϕ is the key point).

Proof. I'll sketch the start of it in class. The rest is an exercise, or see the movie "It's my turn". \Box

2. Tor and Ext

We have so far seen several functors which are left or right exact, but not exact. For instance

- \circ Hom_R $(M, _)$ is left exact
- \circ Hom_R(__, N) is left exact (although contravariant)
- $\circ \otimes$ is right exact

It turns out there is a nice way to handle all these failures of exactness and many others. Through the use of derived functors.

First a formality.

Definition 2.1. Suppose that $B^{\bullet} = \ldots \to B^{-1} \to B^0 \to B^1 \to B^2 \to \ldots$ is a complex (ie, $\ker(B_i \to B_{i+1}) \supseteq \operatorname{im}(B_{i-1} \to B_i)$). We define the *i*th cohomology of B^{\bullet} to be

$$\mathbf{h}^{i}(B^{\bullet}) = \ker(B_{i} \longrightarrow B_{i+1}) / \operatorname{im}(B_{i-1} \longrightarrow B_{i})$$

2.1. **Tor.**

Definition 2.2 (Projective resolutions). Suppose R is a ring and M is an R-module. A projective resolution of M is a series of projective (ie free) modules P_i , i = 0, -1, -2, ... and maps

$$\dots \xrightarrow{f_n} P^{-n} \xrightarrow{f_{n-1}} P^{-n+1} \xrightarrow{f_{n-2}} \dots \xrightarrow{f_2} P^{-2} \xrightarrow{f_1} P^{-1} \xrightarrow{f_0} P^0 \to M \to 0$$

making the above sequence exact. Such a sequence could be infinite. Since every module is a quotient of a free (and hence projective) module, every module has a projective resolution (although not a unique one).

Definition 2.3 (Tor). Suppose that $P^{\bullet} \to M$ is a projective resolution of M. Note that for any module C, $P^{\bullet} \otimes C$ is a complex. We define $\operatorname{Tor}_i(M,C)$ to be $\mathbf{h}^i(P^{\bullet} \otimes C)$. It is not obvious that this is independent of the choice of projective resolution, but it is true.

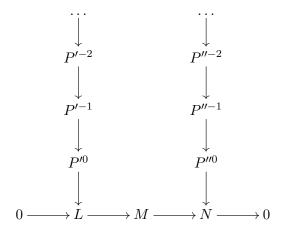
It is easy to see that:

Lemma 2.4. $\operatorname{Tor}_0(M,C) \cong M \otimes C$. Furthermore, if M is projective then $\operatorname{Tor}_i(M,C) = 0$ for all i > 0.

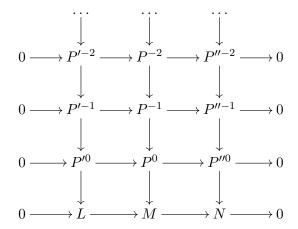
One other fact that is useful, but which we won't prove is that

Lemma 2.5. $\operatorname{Tor}_i(M,C) \cong \operatorname{Tor}_i(C,M)$.

Now suppose that $0 \to L \to M \to N \to 0$ is a short exact sequence. We form a projective resolutions of L and N to form the following:



We set $P_i = P_i' \oplus P_i''$ with the canonical short exact sequences $0 \to P_i' \to P_i \to P_i'' \to 0$. We claim that these combine to form a commutative diagram



where the columns form projective resolutions. This is pretty easy but care must be taken with the details here. Now apply the functor $\otimes_R C$ for some module C to the resolutions $P^{\bullet} = \dots P^2 \to P^1 \to P^0$ (likewise with P'^{\bullet} and P''^{\bullet}). We obtain

$$0 \longrightarrow P'^{-2} \otimes C \longrightarrow P^{-2} \otimes C \longrightarrow P''^{-2} \otimes C \longrightarrow 0$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$0 \longrightarrow P'^{-1} \otimes C \longrightarrow P^{-1} \otimes C \longrightarrow P''^{-1} \otimes C \longrightarrow 0$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$0 \longrightarrow P'^{0} \otimes C \longrightarrow P^{0} \otimes C \longrightarrow P''^{0} \otimes C \longrightarrow 0$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0$$

We now apply the snake lemma, first to the diagram

$$P'^{-1} \otimes C/\operatorname{im}(P'^{-2} \otimes C) \longrightarrow P^{-1} \otimes C/\operatorname{im}(P^{-2} \otimes C) \longrightarrow P''^{-1} \otimes C/\operatorname{im}(P''^{-2} \otimes C) \longrightarrow 0$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$0 \longrightarrow P'^{0} \otimes C \longrightarrow P'^{0} \otimes C \longrightarrow P''^{0} \otimes C \longrightarrow 0$$

The cokernel below the bottom row is simply

$$L \otimes C \longrightarrow M \otimes C \longrightarrow N \otimes C \longrightarrow 0$$

but this snakes up and connects with the kernels above the top row, which are

$$\operatorname{Tor}_1(L,C) \longrightarrow \operatorname{Tor}_1(M,C) \longrightarrow \operatorname{Tor}_1(N,C)$$

connecting these we get a long exact sequence

$$\operatorname{Tor}_1(L,C) \to \operatorname{Tor}_1(M,C) \to \operatorname{Tor}_1(N,C) \to L \otimes C \to M \otimes C \to N \otimes C \to 0.$$

But we don't stop now. We next consider the diagram:

$$P'^{-2} \otimes C/\operatorname{im}(P'^{-3} \otimes C) \longrightarrow P^{-2} \otimes C/\operatorname{im}(P^{-3} \otimes C) \longrightarrow P''^{-2} \otimes C/\operatorname{im}(P''^{-3} \otimes C) \longrightarrow 0$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$0 \longrightarrow \ker(P'^{-1} \otimes C \longrightarrow P'^{0} \otimes C) \longrightarrow \ker(P^{-1} \otimes C \longrightarrow P^{0} \otimes C) \longrightarrow \ker(P''^{-1} \otimes C \longrightarrow P''^{0} \otimes C)$$

applying the snake lemma again (and again) gets us to the long exact sequence

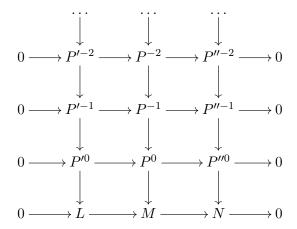
2.2. **Ext.** We first consider the functor $\operatorname{Hom}_R(\underline{\hspace{0.4cm}},C)$.

Definition 2.6. If P^{\bullet} is a projective resolution of M, then we define $\operatorname{Ext}^{i}(M,C)$ to be $\mathbf{h}^{i}(\operatorname{Hom}_{R}(P^{\bullet},C))$, the ith cohomology of the complex $\operatorname{Hom}_{R}(P^{\bullet},C)$.

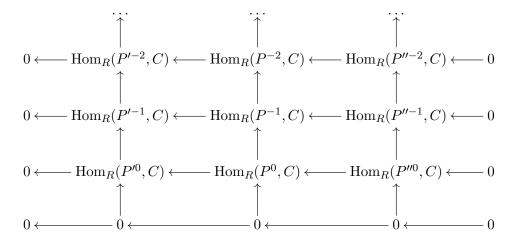
Given a short exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

as above, we again form projective resolutions



and apply the functor $\operatorname{Hom}_R(\underline{\hspace{1em}},C)$ to the projective resolutions to obtain:



Applying the same snake lemma formalisms again, we note that we have diagrams

$$\ker \left(\begin{array}{c} \operatorname{Hom}_R(P'^{-i-1},C) \\ \to \\ \operatorname{Hom}_R(P'^{-i-2},C) \end{array} \right) \longleftarrow \ker \left(\begin{array}{c} \operatorname{Hom}_R(P^{-i-1},C) \\ \to \\ \operatorname{Hom}_R(P^{-i-2},C) \end{array} \right) \longleftarrow \ker \left(\begin{array}{c} \operatorname{Hom}_R(P''^{-i-1},C) \\ \to \\ \operatorname{Hom}_R(P''^{-i-2},C) \end{array} \right) \longleftarrow 0$$

$$\uparrow \qquad \qquad \uparrow \qquad \qquad \downarrow \qquad \qquad \uparrow \qquad \qquad \downarrow \qquad \qquad \uparrow \qquad \qquad \downarrow \qquad$$

The snake lemma yields the following long exact sequence.

$$\begin{array}{cccccc} 0 \longrightarrow & \operatorname{Hom}_R(N,C) & \longrightarrow & \operatorname{Hom}_R(M,C) & \longrightarrow & \operatorname{Hom}_R(L,C) \\ \longrightarrow & \operatorname{Ext}^1_R(N,C) & \longrightarrow & \operatorname{Ext}^1_R(M,C) & \longrightarrow & \operatorname{Ext}^1_R(L,C) \\ \longrightarrow & \operatorname{Ext}^2_R(N,C) & \longrightarrow & \dots \end{array}$$

However, there isn't just one Ext functor... We also have $\operatorname{Hom}_R(B,\underline{\hspace{1em}})$. Projective resolutions just aren't good enough any more. We need

Definition 2.7 (Injective resolutions). Suppose that M is a module. We say that

$$\left(0 \to M \to I^{\bullet}\right) = \left(0 \to M \to I^{0} \to I^{1} \to I^{2} \to \right)$$

is an injective resolution if each I^i is an injective module and the above sequence is a long exact sequence. It is a non-trivial fact that injective resolutions exist (to show it, it is enough to show that for every module N, there is an injective module and an injection $N \hookrightarrow I$).

Definition 2.8. Fix I^{\bullet} to be an injective resolution of a module M and let $\operatorname{Hom}_{R}(B, I^{\bullet})$ be the corresponding complex. Then we define $\operatorname{Ext}_{R}^{i}(B, M)$ to be $\mathbf{h}^{i}(\operatorname{Hom}_{R}(B, I^{\bullet}))$.

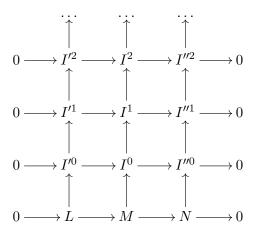
There are a couple key facts we won't prove.

- This Ext is also independent of the choice of injective resolution.
- o This Ext agrees with the other Ext we defined (which is really useful!) In other words

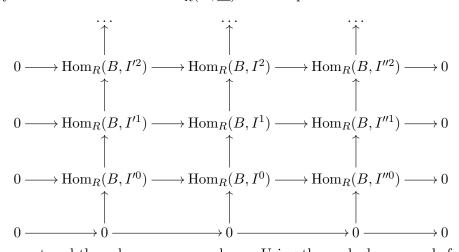
$$\mathbf{h}^i(\operatorname{Hom}_R(B, I^{\bullet})) \cong \mathbf{h}^i(\operatorname{Hom}_R(P^{\bullet}, M))$$

where I^{\bullet} is an injective resolution of M and P^{\bullet} is a projective resolution of B.

Again, given $0 \to L \to M \to N \to 0$ we can form injective resolutions of L and N and take the direct sum to get an injective resolution of M and so have



We can apply the covariant functor $\operatorname{Hom}_R(B,\underline{\hspace{0.1cm}})$ to the I parts and obtain:



The rows are exact and the columns are complexes. Using the snake lemma as before gives us a long exact sequence