HW #7 – MATH 6310
FALL 2017

DUE: FRIDAY, NOVEMBER 10TH

(1) What are the prime ideals in the Gaussian integers $\mathbb{Z}[i]$? (You may use without proof that $\mathbb{Z}[i]$ is a Euclidean domain or PID).
(2) Prove Eisenstein's irreducibility criterion. If $f = a_0 + a_1x + \cdots + a_nx^n \in \mathbb{Z}[x]$ and there exists a prime p such that $p|a_i$ for $i \leq n-1$, $p \not| a_n$ and $p^2 \not| a_0$, then f is irreducible in $\mathbb{Q}[x]$.
(3) Use Eisenstein's criterion to prove that if p is prime in $\mathbb{Z}$ then $x^{p-1} + x^{p-2} + \cdots + 1 = (x^p - 1)/(x-1)$ is irreducible in $\mathbb{Q}[x]$.
(4) Prove that if D is a domain which is not a field, that $D[x]$ is not a PID.
(5) Determine $\mathrm{End}(\mathbb{Q})$ where $\mathbb{Q}$ is an Abelian group under addition.
(6) Let M be a finite nonzero Abelian group. When can M be made into a left $\mathbb{Q}$-module?
(7) Prove that for any ring R and any R-module M, $\mathrm{Hom}_R(R, M) \cong M$.
(8) A left ideal $I \subseteq R$ is called *maximal* if $R \neq I$ and there are no left ideals I' such that $I \subsetneq I' \subsetneq R$. Show that a left R-module M is irreducible if and only if $M \cong R/I$ where I is a maximal left ideal of R.