HW #5 – MATH 6310
FALL 2017

DUE: FRIDAY, OCTOBER 20TH

(1) Show that $\sqrt{3} \notin \mathbb{Q}[\sqrt{2}]$ and that the real numbers $1, \sqrt{2}, \sqrt{3}, \sqrt{6}$ are linearly independent over $\mathbb{Q}$. Show that $\alpha = \sqrt{2} + \sqrt{3}$ is algebraic and determine an ideal $I \subseteq \mathbb{Q}[x]$ such that $\mathbb{Q}[\alpha] \cong \mathbb{Q}[x]/I$.

(2) Suppose that k is a field and that $f = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in k[x] =: R$. Let I denote the ideal $I = (f) = fR$. Consider the quotient R/I with $\overline{x} := x + I \in R/I$. Show that every element of R/I can be written uniquely in the form:

$$b_{n-1}\overline{x}^{n-1} + \cdots + b_1\overline{x}^1 + b_0.$$

where $b_i \in k$.

(3) Take $k = \mathbb{Q}$ and $f = x^3 + 3x - 2$ as in the previous exercise. Show that R/I is a field. Then write the element

$$(\overline{x}^2 - \overline{x} + 4)^{-1}$$

in the form guaranteed in the previous exercise.

(4) Show that $x^3 - x$ has 6 roots in $\mathbb{Z}/6\mathbb{Z}$.