SOME DERIVED CATEGORIES AND FUNCTORS

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1. A crash course in using derived categories

In a nutshell, taking Hom, Ext and $\Gamma(X, \_\_)$ is very useful, but dealing with individual cohomology groups can be a hassle.

Solution: Deal with the complexes instead!

Definition 1.1. A *complex* of R-modules (or $\mathscr{O}_X$-modules if you prefer) is a collection of $\{C^n\}_{n\in\mathbb{Z}}$ of R-modules plus maps $d^n : C^n \longrightarrow C^{n+1}$ such that $d^{i+1} \circ d^i = 0$.

$$\ldots \xrightarrow{d^{-2}} C^{-1} \xrightarrow{d^{-1}} C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \xrightarrow{d^2} \ldots \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \xrightarrow{d^{n+1}} \ldots$$

A complex is *bounded below* if $C^i = 0$ for $i \ll 0$, it is *bounded above* if $C^i = 0$ for $i \gg 0$, and it is *bounded* if $C^i = 0$ for $|i| \gg 0$.

Remark 1.2. In a *chain complex*, the differentials take C_i to C_{i-1}, we will deal exclusively with complexes however.

There are some problems. The category of complexes isn't quite right, so we fix it. We only consider morphisms of complexes up to homotopy equivalence (two maps of complexes are homotopic if their difference is null homotopic), and we *declare* two complexes to be isomorphic if there is a map between them which gives us an isomorphism on cohomology (formally add an inverse map to our category, this is just like how we formally add inverses to rings when localizing, and so this procedure is called localization of categories).

Examples 1.3. Here are some examples you hopefully all are familiar with.

(a) Given a module M, we view it as a complex by considering it in degree zero (all the differentials of the complex are zero) and all the other terms in the complex.
(b) Given any complex $C^{\bullet}$, (like a modules viewed as a complex as above), we can form another complex by shifting the first complex $C[n]^{\bullet}$. This is the complex where $(C[n])^i = C^{i+n}$ and where the differentials are shifted likewise and multiplied by $(-1)^n$. Note this shifts the complex n spots to *the left.*
(c) Given a module M, and a projective resolution

$$\ldots \longrightarrow P^{-n} \longrightarrow \ldots \longrightarrow P^{-2} \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow M \longrightarrow 0$$

it is easy to see that there is a map $P^\bullet \to M$ and this is a map of complexes in the sense that M is a complex via (a).

$$\begin{tikzcd} \dots \arrow[r] & P^{-n} \arrow[r] \arrow[d] & \dots \arrow[r] & P^{-2} \arrow[r] \arrow[d] & P^{-1} \arrow[r] \arrow[d] & P^0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \dots \\ \dots \arrow[r] & 0 \arrow[r] & \dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0 \arrow[r] & \dots \end{tikzcd}$$

This map is a quasi-isomorphism (an isomorphism in the derived category).

(d) Given a module M and an injective resolution

$$0 \to M \to I^0 \to I^1 \to I^2 \to \dots$$

we get a map of complexes $M \to I^\bullet$

$$\begin{tikzcd} \dots \arrow[r] & 0 \arrow[r] & M \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \dots \\ \dots \arrow[r] & 0 \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & I^2 \arrow[r] & \dots \end{tikzcd}$$

which is also a quasi-isomorphism (again viewing M as a complex via (a)).

(e) Given two modules M and N, we form $\mathbf{R}\operatorname{Hom}_R(M,N)$. This is the complex whose cohomologies are the $\operatorname{Ext}^i(M,N)$. It is computed by either taking a projective resolution of M or an injective resolution of N. Note that while you get different complexes in either of those cases, it turns out the resulting objects in the derived category are isomorphic (in the derived category).

(f) Given two modules M and N, we form $M \otimes^L_R N$, the cohomologies of this complex are the $\operatorname{Tor}^i_R(M,N)$. It is obtained by taking a projective resolution of M or N.

(g) Note that not every quasi-isomorphism between complexes is invertible. Indeed, consider

$$\begin{tikzcd} \dots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "\times 2"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \dots \\ \dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \arrow[r] & \dots \end{tikzcd}$$

This obviously induces a quasi-isomorphism since the top row is a projective resolution of the bottom, but the map of complexes is not invertible. In the *derived* category, we formally adjoin an inverse morphism.

(h) Not every pair of complexes with isomorphic cohomologies are quasi-isomorphic, indeed consider the complexes

$$\dots \to 0 \to \mathbb{C}[x,y] \xrightarrow{0} \mathbb{C} \to 0 \to \dots$$

and

$$\dots \to 0 \to \mathbb{C}[x,y]^{\oplus 2} \xrightarrow{[x,y]} \mathbb{C}[x,y] \to 0 \to \dots.$$

It is easy to see that they have isomorphic cohomologies. Some discussion of the fact that these complexes are not quasi-isomorphic can be found in the responses to this question on `math.stackexchange`, [hf].

Remark 1.4. There are different ways to enumerate things, but complexes have maps that go left to right. Thus a projective resolution of a module M has entries only in *negative* degrees. Thus when I write $\mathrm{Tor}^i_R(M,N)$ above, the i that can have interesting cohomology are the $i \le 0$.

Definition 1.5. The *derived category of R-modules* denoted $D(R)$ is the category of complexes with morphisms defined up to homotopy and with quasi-isomorphisms formally inverted. Likewise $D(X)$ is the category of complexes with morphisms defined up to homotopy and with quasi-isomorphisms formally inverted.

If we look at the full subcategory of complexes bounded above, and then construct the derived category as above, the result is denoted by $D^-(R)$. From bounded below complexes, we construct $D^+(R)$. Finally, if the complexes are bounded on both sides the result is denoted by $D^b(R)$. If we restrict to complexes of $\mathscr{O}_X$-modules whose cohomology is coherent, we might denote that by $D_{\mathrm{coh}}(X)$. Likewise for complexes with quasi-coherent cohomology, we might use $D_{\mathrm{qcoh}}(X)$.

2. Triangulated categories

Derived categories are *not* an Abelian category, short exact sequences don't exist, but we have something almost as good, exact triangles. In particular, the derived category is a triangulated category.

Remark 2.1. The notation from the following axioms is taken from [Wei94] (you can find different notation on for instance Wikipedia).

Definition 2.2 (Triangulated categories)**.** A *triangulated category* is an additive[1] category with a fixed automorphism T equipped with a distinguished set of triangles and satisfying a set of axioms (below). A *triangle* is an ordered triple of objects (A, B, C) and morphism $\alpha : A \longrightarrow B$, $\beta : B \longrightarrow C$, $\gamma : C \longrightarrow T(A)$,

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} T(A).$$

A *morphism of triangles* $(A, B, C, \alpha, \beta, \gamma) \longrightarrow (A', B', C', \alpha', \beta', \gamma')$ is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"'] & B \arrow[r, "\beta"] \arrow[d, "g"'] & C \arrow[r, "\gamma"] \arrow[d, "h"'] & T(A) \arrow[d, "T(f)"'] \\ A' \arrow[r, "\alpha'"'] & B' \arrow[r, "\beta'"'] & C' \arrow[r, "\gamma'"'] & T(A') \end{tikzcd}$$

We now list the required axioms to make a triangulated category.

(a) The triangle $A \longrightarrow A \xrightarrow{0} 0 \xrightarrow{0} T(A)$ is one of the distinguished triangles.

[1]Hom sets are Abelian groups and composition is bilinear.

(b) A triangle isomorphic to one of the distinguished triangles is distinguished.
(c) Any morphism $A \longrightarrow B$ can be embedded into one of the distinguished triangles $A \longrightarrow B \longrightarrow C \longrightarrow T(A)$. In our case, the object C is normally the cone.
(d) Given any distinguished triangle $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} T(A)$, then both

$$B \xrightarrow{\beta} C \xrightarrow{\gamma} T(A) \xrightarrow{-T(\alpha)} T(B)$$

and

$$T^{-1}C \xrightarrow{-T^{-1}(\gamma)} A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

are also distinguished.
(e) Given distinguished triangles with maps between them as pictured below, so that the left square commutes,

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"'] & B \arrow[r, "\beta"] \arrow[d, "g"'] & C \arrow[r, "\gamma"] \arrow[d, dotted, "\exists h"'] & T(A) \arrow[d, "T(f)"'] \\ A' \arrow[r, "\alpha'"'] & B' \arrow[r, "\beta'"'] & C' \arrow[r, "\gamma'"'] & T(A') \end{tikzcd}$$

then the dotted arrow also exists and we obtain a morphism of triangles.
(f) We finally come to the feared *octahedral axiom*. Given objects A, B, C, A', B', C' and three distinguished triangles:

$$A \xrightarrow{u} B \xrightarrow{j} C' \xrightarrow{\partial} T(A)$$

$$B \xrightarrow{v} C \xrightarrow{x} A' \xrightarrow{i} T(B)$$

$$A \xrightarrow{v \circ u} C \xrightarrow{y} B' \xrightarrow{\delta} T(A)$$

then there exists a fourth triangle

$$C' \xrightarrow{f} B' \xrightarrow{g} A' \xrightarrow{(T(j)) \circ i} T(C')$$

so that we have

$$\partial = \delta \circ f, x = g \circ y, y \circ v = f \circ j, u \circ \delta = i \circ g.$$

These can be turned into a nice octagon (with these equalities being commuting faces) that I am too lazy to LaTeX.

Remark 2.3. It is much easier to remember the octahedral axiom (without the compatibilities at least) with the following diagram.

$$\begin{tikzcd}
 & & C' \arrow[ddr, dotted, "\exists f"] & & \\
 & B \arrow[ur, "j"] \arrow[dr, "v"] & & & \\
A \arrow[ur, "u"] \arrow[rr, "v \circ u"'] & & C \arrow[r, "y"] \arrow[drr, "x"'] & B' \arrow[dr, dotted, "\exists g"] & \\
 & & & & A'
\end{tikzcd}$$

Any of the derived categories we have discussed are triangulated categories with $T(\bullet) = \bullet[1]$. The main point is if we have a morphism of complexes, $A^\bullet \xrightarrow{\alpha} B^\bullet$, then we can always take the cone $C(\alpha)^\bullet = A[1]^\bullet \oplus B^\bullet$ with differential

$$C^i = A^{i+1} \oplus B^i \xrightarrow{-d_A^{i+1}, \alpha^i + d_B^i} A^{i+2} \oplus B^{i+1}$$

Exercise 2.1. Verify that this really is a complex.

Exercise 2.2. Suppose that $0 \longrightarrow A^\bullet \xrightarrow{\alpha} B^\bullet \xrightarrow{\beta} D^\bullet \longrightarrow 0$ is an exact sequence of complexes. Show that $D^\bullet$ is quasi-isomorphic to $C(\alpha)^\bullet$.

Then we have $A^\bullet \xrightarrow{\alpha} B^\bullet \xrightarrow{\beta} C(\alpha)^\bullet \gamma A[1]^\bullet$ a distinguished triangle where β and γ are given by maps to and projecting from the direct summands that make up $C(\alpha)^\bullet$. Note that morphisms in the derived category are more complicated than maps between complexes (since we might have formally inverted some quasi-isomorphisms) but this still is enough for our purposes since the cone of a quasi-isomorphism is exact.

Fact 2.4. Given a triangle $A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow A[1]^\bullet$ in the derived category of R-modules, taking cohomology yields a long exact sequence

$$\ldots \longrightarrow h^{i-1}(C^\bullet) \longrightarrow h^i(A^\bullet) \longrightarrow h^i(B^\bullet) \longrightarrow h^i(C^\bullet) \longrightarrow h^{i+1}(A^\bullet) \longrightarrow \ldots$$

Exercise 2.3. Verify that fact.

Exercise 2.4. Suppose that

$$A^\bullet \xrightarrow{\alpha} B^\bullet \xrightarrow{\beta} C^\bullet \xrightarrow{0} T(A^\bullet)$$

is a distinguished triangle in $D(R)$. Show that $B^\bullet \simeq_{\text{qis}} A^\bullet \oplus B^\bullet$ compatibly so that α and β are identified with the canonical inclusion and projections.

In particular, show that there exist maps $p : B^\bullet \longrightarrow A^\bullet$ and $s : C^\bullet \longrightarrow B^\bullet$ so that $p \circ \alpha$ is the identity on $A^\bullet$ and that $\beta \circ s$ is the identity on $C^\bullet$.

3. Common functors on our derived categories

Suppose we are forming the derived category $D(R)$ of the category of R-modules for some ring R. We have lots of functors we like to apply to R-modules, notably Hom and $\otimes$ but also things like Γ_I (the submodule of things killed by a power of I). Associated to any of these functors we get derived functors, as follows.

Derived functors are functors between triangulated categories which preserve the triangulation structure (ie, send triangles to triangles and commute with the $T(\bullet)/[\bullet]$ operation) and which satisfy a certain universal property which we won't need too much (see for example [Wei94, Section 10.5] for details). The point for us is that derived functors exist for the functors we care about.

Lemma 3.1. [Wei94, Corollary 10.5.7] *Suppose $F : \mathscr{M}\mathrm{od}(R) \longrightarrow \mathscr{M}\mathrm{od}(S)$ is an additive functor which takes R-modules to S-modules. Then the right derived functors $\mathbf{R}F : D^+(R) \longrightarrow D(S)$ are morphisms between triangulated categories and can be computed by $\mathbf{R}F(C^\bullet) = F(I^\bullet)$ where $I^\bullet$ is a complex of injectives quasi-isomorphic to $C^\bullet$. In particular, $h^i\mathbf{R}F(C^\bullet) = \mathbb{R}^i F(C^\bullet)$.*

Likewise, the left derived functors $\mathbf{L}F : D^-(R) \longrightarrow D(S)$ can be computed by $\mathbf{R}F(C^\bullet) = F(P^\bullet)$ where $P^\bullet$ is a complex of projectives quasi-isomorphic to $C^\bullet$.

If you are really Hom'ing or tensoring two complexes together, you typically need to actually compute this by forming the associated double complex and then taking the total complex, see for example page 8 of [Wei94]. For example if $M^\bullet$ and $N^\bullet$ are complexes made up of projectives (or at least one of them is), then the total complex of the double complex represents the object $M^\bullet \otimes^{\mathbf{L}}_R N^\bullet$.

Notably, we have

- $\mathbf{R}\operatorname{Hom}_R(A^\bullet, B^\bullet)$ can be computed by taking a complex of projectives quasi-isomorphic to $A^\bullet \in D^-(R)$ or a complex of injectives quasi-isomorphic to $B^\bullet \in D^+(R)$. Note if A, B are modules, then $h^i\mathbf{R}\operatorname{Hom}(A, B) = \operatorname{Ext}^i(A, B)$
- $A^\bullet \otimes^{\mathbf{L}}_R B^\bullet$ can be computed by taking a complex of projectives quasi-isomorphic to either $A^\bullet$ or $B^\bullet$ in $D^-(R)$. Note if A, B are modules, then $h^{-i}(A \otimes^{\mathbf{L}}_R B) = \operatorname{Tor}^R_i(A, B)$.
- For any ideal $I \subseteq R$, recall that $\Gamma_I(M) = \{m \in M \mid I^n m = 0 \text{ for some } n \gg 0\}$. Then $\mathbf{R}\Gamma_I(A^\bullet)$ is computed by finding a complex of injectives quasi-isomorphic to $A^\bullet \in D^+(R)$. Note that if A is a module, then $h^i\mathbf{R}\Gamma_I(A) = H^i_I(A)$ is just local cohomology.
- For any sheaf of $\mathscr{O}_X$-module $\mathscr{F}$, we form $\mathbf{R}\Gamma(X, \mathscr{F})$ by computing $\Gamma(X, \mathscr{I}^\bullet)$ where $\mathscr{I}^\bullet$ is an injective resolution of $\mathscr{F}$. Likewise if $\mathscr{F} \in D^+(X)$ is a complex.
- If $f : X \longrightarrow Y$ is a morphism of schemes, and $\mathscr{G}^\bullet \in D^+(X)$, we compute $\mathbf{R}f_*\mathscr{G}^\bullet$ by finding $f_*\mathscr{I}^\bullet$ where $\mathscr{I}^\bullet$ is a complex of injectives quasi-isomorphic to $\mathscr{G}^\bullet$.
- If $f : X \longrightarrow Y$ is a morphism of schemes, and $\mathscr{G}^\bullet \in D^-(X)$, then we define the left derived functor $\mathbf{L}f^*\mathscr{G}^\bullet$ by taking a flat resolution of $\mathscr{G}^\bullet$.

The rest of the chapter is devoted to how these functors play with each other.

Theorem 3.2 (Composition of derived functors, left-exact case)**.** *Given left exact functors $G : \mathscr{M}\mathrm{od}(R) \longrightarrow \mathscr{M}\mathrm{od}(S)$ and $F : \mathscr{M}\mathrm{od}(S) \longrightarrow \mathscr{M}\mathrm{od}(T)$ (or suitable Abelian categories with enough injectives), and suppose that G sends injective objects to F-acyclic objects, then $\mathbf{R}F \circ \mathbf{R}G \cong \mathbf{R}(F \circ G)$ as functors from $D^+(R) \longrightarrow D^+(T)$.*

For a more general statement, see [Wei94, Theorem 10.8.2]. Things that imply the above, make a lot of the formulas we already know relating Hom and $\otimes$ and other functors hold in the derived category as well.

We list some of them here without proof, see for example [Wei94, Section 10.8] or [Har66, II, Section 5].

Proposition 3.3. *The following hold:*

(a) Let $f : R \longrightarrow S$ be a map of rings with functors $f^* : \mathscr{M}\mathrm{od}(R) \longrightarrow \mathscr{M}\mathrm{od}(S)$ defined by $f^*(M) = M \otimes_R S$ and $f_* : \mathscr{M}\mathrm{od}(S) \longrightarrow \mathscr{M}\mathrm{od}(R)$ defined by f_*N is N viewed as an R-module via restriction of scalars. Then for $A^\bullet \in D^-(R), B^\bullet \in D^-(S)$ we have

$$\mathbf{L}f^*(A^\bullet) \otimes^{\mathbf{L}}_S B^\bullet \cong A^\bullet \otimes^{\mathbf{L}}_R f_* B^\bullet .$$

This is a special case (the affine case) of the derived projection formula you might have seen in your algebraic geometry class.

(b) For $A^\bullet, B^\bullet \in D^-(R)$ and $C^\bullet \in D^+(R)$, we have

$$\mathbf{R}\operatorname{Hom}_R(A^\bullet, \mathbf{R}\operatorname{Hom}_R(B^\bullet, C^\bullet)) \cong \mathbf{R}\operatorname{Hom}_R(A^\bullet \otimes^{\mathbf{L}}_R B^\bullet, C^\bullet)$$

in $D^+(R)$. This is just derived Hom, $\otimes$ *adjointness.*

(c) For $A^\bullet \in D^-(R)$ and $B^\bullet \in D^+(R)$ and $C^\bullet \in D^b(R)$ of bounded Tor*-dimension (for example, bounded projective dimension), ie the projective resolution of anything in a regular ring, then*

$$\mathbf{R}\operatorname{Hom}_R(A^\bullet, B^\bullet) \otimes^{\mathbf{L}}_R C^\bullet \cong \mathbf{R}\operatorname{Hom}_R(A^\bullet, B^\bullet \otimes^{\mathbf{L}}_R C^\bullet).$$

(d) Consider two ideals $I, J \subseteq R$ in a Noetherian ring. Then $\Gamma_I \circ \Gamma_J = \Gamma_{I+J} = \Gamma_{\sqrt{I+J}}$ as is easily checked. Next suppose that M is an injective module, we want to show that $\Gamma_J(M)$ is Γ_I-acyclic. This is normally done by showing that $\Gamma_J(M)$ is flasque and I won't reproduce it here. Thus we have that

$$\mathbf{R}\Gamma_I \circ \mathbf{R}\Gamma_J = \mathbf{R}\Gamma_{I+J}$$

In the case case that $I \supseteq J$ we see that

$$\mathbf{R}\Gamma_I \circ \mathbf{R}\Gamma_J = \mathbf{R}\Gamma_I.$$

(e) If $f : X \longrightarrow Y$ is a morphism of schemes, then there is a natural isomorphism $\mathbf{R}\Gamma(X, \mathscr{F}^\bullet) \xrightarrow{\sim} \mathbf{R}\Gamma(Y, \mathbf{R}f_*\mathscr{F}^\bullet)$ for $\mathscr{F}^\bullet \in D^+(X)$. This is the analog of the fact that $\Gamma(Y, f_*\mathscr{F}) = \Gamma(X, \mathscr{F})$.

(f) If X is a scheme, there is a natural isomorphism of functors $\mathbf{R}\operatorname{Hom}^\bullet_X(F^\bullet, G^\bullet) \xrightarrow{\sim} \mathbf{R}\Gamma(X, \mathbf{R}\mathscr{H}\mathrm{om}^\bullet_{\mathscr{O}_X}(F^\bullet, G^\bullet))$ for $\mathscr{F}^\bullet \in D^-(X)$ and $\mathscr{G}^\bullet \in D^+(X)$. This is the derived analog of the fact that the global sections of $\mathscr{H}\mathrm{om}$ are Hom.

(g) *If $f : X \longrightarrow Y$ is a morphism of Noetherian schemes of finite dimension such that the inverse image of an open set is quasi-compact (this is very mild). Then there is a natural functorial isomorphism*

$$(\mathbf{R}f_* \mathscr{F}^\bullet) \otimes^{\mathbf{L}}_{\mathscr{O}_Y} \mathscr{G}^\bullet \xrightarrow{\sim} \mathbf{R}f_*(\mathscr{F}^\bullet \otimes_{\mathscr{O}_X} \mathbf{L}f^* \mathscr{G}^\bullet)$$

for $F^\bullet \in D^-(X)$ and $G^\bullet \in D^-_{\mathrm{qcoh}}(Y)$. This is the derived version of the projection formula.

(h) *Suppose $f : X \longrightarrow Y$ is a morphism of schemes and $\mathscr{F}^\bullet \in D^-_{\mathrm{coh}}(Y)$. Then there is a natural functorial morphism $F^\bullet \longrightarrow \mathbf{R}f_* \mathbf{L}f^* F^\bullet$ which gives a natural functorial isomorphism $\mathbf{R}f_* \mathscr{H}\mathrm{om}^\bullet_{\mathscr{O}_X}(\mathbf{L}f^* \mathscr{F}^\bullet, \mathscr{G}^\bullet) \xrightarrow{\sim} \mathbf{R} \mathscr{H}\mathrm{om}^\bullet_{\mathscr{O}_Y}(\mathscr{F}^\bullet, \mathbf{R}f_* \mathscr{G}^\bullet)$. This is the derived version of the adjointness between f_* and f^*.*

(i) *Suppose X is a Noetherian scheme and that every coherent sheaf is a quotient of a locally free finite rank sheaf (for instance, if X is quasi-projective). Then there is a natural functorial isomorphism*

$$\mathbf{R} \mathscr{H}\mathrm{om}^\bullet (\mathscr{F}^\bullet, \mathbf{R} \mathscr{H}\mathrm{om}^\bullet (\mathscr{G}^\bullet, \mathscr{H}^\bullet)) \xrightarrow{\sim} \mathbf{R} \mathscr{H}\mathrm{om}^\bullet (\mathscr{F}^\bullet \otimes^{\mathbf{L}} \mathscr{H}^\bullet)$$

for $\mathscr{F}^\bullet, \mathscr{G}^\bullet \in D^-_{\mathrm{coh}}(X)$ and $\mathscr{H}^\bullet \in D^+(X)$. This is the analog of sheafy Hom-tensor adjointness.

References

[Har66] R. Hartshorne: *Residues and duality*, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20, Springer-Verlag, Berlin, 1966. MR0222093 (36 #5145)

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