WORKSHEET ON THE NULLSTELLENSATZ FOR ARBITRARY RINGS

MATH 538
FALL 2013

Suppose R is a ring.

Definition 0.1 (Residue fields, vanishing functions)**.** For each prime $P \in \operatorname{Spec} R$, we define the *residue field* $k(P)$ to be the field of fractions of R/P. We say that $f \in R$ *vanishes at* P if the image of f is zero in $k(P)$ (this is clearly the same as asking that $f \in P$), in which case we write $f(P) = 0$. More generally $f(P)$ denotes the image of f in $k(P)$. Given any subset $Z \subseteq \operatorname{Spec} R$, we define

$$I(Z) = \{f \in R \mid f(P) = 0 \text{ for all } P \in \operatorname{Spec} R\}$$

We will prove a version of the Nullstellensatz in this generality, where everything is much easier.

1. Show that $I(Z) = R$ if and only if $Z = \emptyset$.

Solution: If $I(Z) = R$, then $1 \in R$ vanishes at every prime of Z. But there are no such primes. Conversely if $Z = \emptyset$, then every element of R vanishes at every element of Z, vacuously.

2. Show that $I(V(J)) = \sqrt{J}$.

Solution: Note first that $f(P) = 0$ if and only if $f \in P$. Recall that $\sqrt{J} = \bigcap_{P \in \operatorname{Spec} R, P \supseteq J} P$ (or if you don't recall it, prove it, it's an easy exercise). On the other hand $I(Z) = \bigcap_{P \in Z} P$ since $f(P) = 0$ if and only if $f \in P$. Thus $I(V(J)) = \bigcap_{P \in V(J)} P$ which is exactly $\sqrt{J}$.

3. Show that $V(J + J') = V(J) \cap V(J')$ and that $V(J \cap J') = V(J) \cup V(J')$.

Solution: For $V(J + J') = V(J) \cap V(J')$ we first observe that $V(J + J') \subseteq V(J), V(J')$ so $\subseteq$ is easy. Now suppose that $P \in V(J) \cap V(J')$. Hence $f(P) = 0$ for all $f \in J$ and all $g \in J'$. But then $(f + g)(P) = 0$ for all $h = f + g \in J + J'$.

For the second statement, again we have $V(J), V(J') \subseteq V(J \cap J')$ so $\supseteq$ is obvious. Now suppose that $P \in V(J \cap J')$. Hence for every $f \in J \cap J'$, $f(P) = 0$ and so $J \cap J' \subseteq P$. But now by an old homework assignment, either $J \subseteq P$ or $J' \subseteq P$. In the first case $P \in V(J)$ and in the second $P \in V(J')$. In either case we are done though.

4. Show that $I(Y \cup Z) = I(Y) \cap I(Z)$ and *if* Y *and* Z *are closed* that $I(Y \cap Z) = \sqrt{I(Y) + I(Z)}$.

Solution: Note $I(Y), I(Z) \supseteq I(Y \cup Z)$ so $\subseteq$ is clear. Choose $f \in I(Y) \cap I(Z)$. Hence f vanishes for each P in Y and each Q in Z and so it vanishes at all points of $Y \cup Z$ as desired.

For the second statement, again clearly $I(Y \cap Z) \supseteq I(Y), I(Z)$ and so $\supseteq$ is clear (since both sides are ideals and the left side is radical). Next since Y and Z are closed, $Y = V(J)$ and $Z = V(J')$ for some ideals J, J' and so $I(Y) = \sqrt{J}$ and $I(Z) = \sqrt{J'}$. Note $Y \cap Z = V(J + J')$ by **3.**. Hence we observe that

$$I(Y \cap Z) = I(V(J + J')) = \sqrt{J + J'}$$

as desired.

Without the hypothesis that Y and Z are closed, the last remark is false. For an example, let $R = \mathbb{C}[x]$ and let Y be the real line (the primes $\langle x - r \rangle$, $r \in \mathbb{R}$) and Z be the imaginary line (the primes $\langle x - ir \rangle$, $r \in \mathbb{R}$). Then $I(Y \cap Z) = I(\{\langle x \rangle\}) = \langle x \rangle$. But individually $I(Y) = I(Z) = R$.