WORKSHEET ON THE NULLSTELLENSATZ FOR ARBITRARY RINGS

MATH 538
FALL 2013

Suppose R is a ring.

Definition 0.1 (Residue fields, vanishing functions)**.** For each prime $P \in \operatorname{Spec} R$, we define the *residue field* $k(P)$ to be the field of fractions of R/P. We say that $f \in R$ *vanishes at* P if the image of f is zero in $k(P)$ (this is clearly the same as asking that $f \in P$), in which case we write $f(P) = 0$. More generally $f(P)$ denotes the image of f in $k(P)$. Given any subset $Z \subseteq \operatorname{Spec} R$, we define

$$I(Z) = \{f \in R \mid f(P) = 0\}$$

We will prove a version of the Nullstellensatz in this generality, where everything is much easier.

1. Show that $I(Z) = R$ if and only if $Z = \emptyset$.

2. Show that $I(V(J)) = \sqrt{J}$.

3. Show that $V(J + J') = V(J) \cap V(J')$ and that $V(J \cap J') = V(J) \cup V(J')$.

4. Show that $I(Y \cup Z) = I(Y) \cap I(Z)$ and that *if* Y *and* Z *are closed* $I(Y \cap Z) = \sqrt{I(Y) + I(Z)}$.