WORKSHEET ON INTEGRAL EXTENSIONS

MATH 538
FALL 2013

Suppose $R \subseteq S$ is an extension of rings.

1. Suppose that R and S are integral domains and $R \subseteq S$ is integral. Show that R is a field if and only if S is a field.

Hint: If R is a field, choose $y \in S$ write an integral equation. Conversely if S is a field choose $x \in R$, note that $1/x \in S$ and use the fact that it is integral over R.

Solution: Suppose R is a field and $y \in S \setminus R$. Then since y is integral over R we can write $y^n + r_{n-1}y^{n-1} + \cdots + r_0 = 0$ for $r_i \in R$, hence y is algebraic over R and $R[y]$ is a field. In particular $y^{-1} \in R[y] \in S$.

Conversely, suppose that S is a field. Choose $x \in R$, then $x \in S$ and so $x^{-1} \in S$. We see that x^{-1} is integral over R and so that $x^{-n} + r_{n-1}x^{-n+1} + \cdots + r_0 = 0 \in S$ for some $r_i \in R$. Multiplying through by x^{n-1} we see that

$$x^{-1} + r_{n-1} + r_{n-2}x + r_{n-3}x^2 + \cdots + r_0 x^{n-1} = 0$$

and hence $x^{-1} \in R$ as desired.

2. Suppose that $R \subseteq S$ is integral and $Q \subseteq S$ is a prime ideal. Show that $Q \cap R$ is maximal if and only if Q is maximal.

Hint: Use **1.**

Solution: Let $P = Q \cap R$. Then we have an injection $R/P \hookrightarrow S/Q$. Then P is maximal if and only if R/P is a field if and only if S/Q is a field (by **1.**) if and only if Q is maximal.

3. Suppose that $R \subseteq S$ is integral (for instance, if S is a finitely generated R-module). Show that $\operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ is surjective.

Hint: Choose a $P \in \operatorname{Spec} R$ then invert $R \setminus P$ in both R and S.

Solution: Choose $P \in \operatorname{Spec} R$ and set $W = R \setminus P$. Consider the induced map $R_P = W^{-1}R \longrightarrow W^{-1}S$. Let Q' be a maximal ideal of $W^{-1}S$ corresponding to $Q \in \operatorname{Spec} S$. Then it is easy to see that $Q' \cap (W^{-1}R)$ corresponds to $Q \cap R$. By **2.** $Q' \cap (W^{-1}R)$ is a maximal ideal. But there is only one such maximal ideal PR_P since R_P is local. Thus $Q' \cap (W^{-1}R) = PR_P$ and so $Q \cap R = P$.

4. Suppose that $R \subseteq S$ is a finite extension (meaning that S is a finitely generated R-module), show that $\operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ is finite-to-1.

Hint: Work as in **3.** but now also mod out both R and S by P (why is $S/(P \cdot S) \neq 0$?).

Solution: Fix $P \in \operatorname{Spec} R$ and set $W = R \setminus P$. The primes of S that have pre-image P is in bijection with the set of primes of $B = W^{-1}\big(S/(P \cdot S)\big)$. But we have a map $k = R_P/P \cdot R_P \hookrightarrow B$. It is easy to see that this is a finite extension of rings. Hence we have reduced the problem to proving the following lemma.

Lemma. *If $k \subseteq B$ is an extension of rings where k is a field, then B has finitely many prime ideals.*

Proof of lemma. We proceed by induction on $\dim_k B$, the base case is obvious. Choose $b \in B \setminus k$ and consider $k[b] \subseteq B$, also a finite extension of k. Obviously $k[b] \cong k[x]/\langle f \rangle$ has finitely many prime ideals, corresponding to the prime factors of f. Hence it is sufficient to show that for each prime $P \in \operatorname{Spec} k[b]$, there are finitely many primes of B lying over P (ie, primes $Q \in \operatorname{Spec} B$ such that $Q \cap k[b] = P$). These correspond to the primes of $B/(P \cdot B)$. Consider $K = k[b]/P \supseteq k$. This is also a finite extension of k and since it is an integral domain, it is a field. Hence it has a single prime ideal. There are two cases.

(a) $K = k$. In that case $P \neq 0$ (since $b \notin k$). Thus consider $k[b]/P \subseteq B/(P \cdot B)$. We see that $\dim_k(B/(P \cdot B)) < \dim_k B$ and so induction implies that $B/(P \cdot B)$ has finitely many prime ideals.

(b) $K \neq k$. In this case $\dim_k B \geq \Big(\dim_K(B/(P \cdot B))\Big) \cdot \Big(\dim_k K\Big)$ but $\dim_k K \geq 2$ and the result follows.

□

5. Suppose that $R \subseteq S$ is integral Suppose that $Q \subseteq Q'$ are prime ideals of S and that $P = Q \cap R = Q' \cap R$. Show that $Q = Q'$.

Hint: Localize appropriately, and look for a chain of maximal ideals.

Solution: This is a homework problem now.

6. [The going up theorem] Suppose that $R \subseteq S$ is integral. If $P_1 \subseteq \ldots \subseteq P_n$ is a chain of prime ideals of R and $Q_1 \subseteq \ldots \subseteq Q_m$ is a chain of prime ideals of R with $m < n$ and $Q_i \cap R = P_i$ for $1 \leq i \leq m$, then there exist $Q_{i+1} \subseteq \ldots \subseteq Q_n$, containing Q_i so that $Q_i \cap R = P_i$ for $1 \leq i \leq n$.

Solution: This is a homework problem now.