WORKSHEET ON INTEGRAL EXTENSIONS

MATH 538
FALL 2013

Suppose $R \subseteq S$ is an extension of rings.

1. Suppose that R and S are integral domains and $R \subseteq S$ is integral. Show that R is a field if and only if S is a field.

Hint: If R is a field, choose $y \in S$ write an integral equation and solve for y. Conversely if S is a field choose $x \in R$, note that $1/x \in S$ and use the fact that it is integral over R.

2. Suppose that $R \subseteq S$ is integral and $Q \subseteq S$ is a prime ideal. Show that $Q \cap R$ is maximal if and only if Q is maximal.

Hint: Use **1.**

3. Suppose that $R \subseteq S$ is integral (for instance, if S is a finitely generated R-module). Show that $\operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ is surjective.

Hint: Choose a $P \in \operatorname{Spec} R$ then invert $R \setminus P$ in both R and S.

4. Suppose that $R \subseteq S$ is a finite extension (meaning that S is a finitely generated R-module), show that $\operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ is finite-to-1.

Hint: Work as in **3.** but now also mod out both R and S by P (why is $S/(P \cdot S) \neq 0$?).

5. Suppose that $R \subseteq S$ is integral Suppose that $Q \subseteq Q'$ are prime ideals of S and that $P = Q \cap R = Q' \cap R$. Show that $Q = Q'$.

Hint: Localize appropriately, and look for a chain of maximal ideals.

6. [The going up theorem] Suppose that $R \subseteq S$ is integral. If $P_1 \subseteq \ldots \subseteq P_n$ is a chain of prime ideals of R and $Q_1 \subseteq \ldots \subseteq Q_m$ is a chain of prime ideals of R with $m < n$ and $Q_i \cap R = P_i$ for $1 \leq i \leq m$, then there exist $Q_{i+1} \subseteq \ldots \subseteq Q_n$, containing Q_i so that $Q_i \cap R = P_i$ for $1 \leq i \leq n$.