HOMEWORK #6 – MATH 538
FALL 2013

DUE THURSDAY, NOVEMBER 21ST

(1) Suppose that M is an R-module and
$$0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \ldots \subsetneq M_n = M$$
is a chain of submodules such that each M_i/M_{i+1} is *simple* (the only submodules are 0 and itself). This is called a *composition series for* M. Show that the length of the composition series is independent and so $n = \lambda(M)$ the length of M.

(2) Suppose that $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ is a short exact sequence. Show that $\lambda(M) = \lambda(L) + \lambda(N)$ where λ is the length of M. Conclude that if $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow \ldots \longrightarrow M_s \longrightarrow 0$ is exact then $\sum (-1)^i \lambda(M_i) = 0$.

(3) Suppose that R is a Noetherian ring, $\mathfrak{a} \subseteq R$ is an ideal and $\widehat{R}$ is the $\mathfrak{a}$-adic completion of R with natural map $h : R \longrightarrow \widehat{R}$. Show that if $x \in R$ is not a zero divisor, then $h(x)$ is also not a zero divisor.

The next problem gives an example where $\widehat{R}$ is *not* an integral domain, even though R is.

(4) Computations of completions.
   (a) Suppose that $R = k[x, y]/\langle y^2 - x^3 - x^2\rangle$. Compute the completion $\widehat{R}$ of R at the maximal ideal $\langle x, y\rangle$. Show that $\widehat{R}$ is not an integral domain even if R is. Find a geometry reason for this.
   (b) Now determine if $R = k[x, y]/\langle y^2 - x^3\rangle$ is a domain after completion at $\langle x, y\rangle$.

(5) Suppose that R is a ring containing a field of characteristic $p > 0$. Let $F : R \longrightarrow S = R$ be the Frobenius morphism. (I use S to help distinguish the target and source).
   (a) Suppose that $W \subseteq R$ is a multiplicative set. Show that $W^{-1}S$ is (abstractly) isomorphic to $W^{-1}R$ and that the the induced map $W^{-1}R \longrightarrow W^{-1}S$ is the Frobenius morphism.
   (b) Now suppose that $\mathfrak{a} \subseteq R$ is an ideal, R is Noetherian, and F makes S into a finitely generated R-module. Show that the $\mathfrak{a}$-adic completion $\widehat{S}$ of S is (abstractly) isomorphic to the $\mathfrak{a}$-adic completion $\widehat{R}$ of R. Show that with this identification, the induced map
$$\widehat{R} \longrightarrow \widehat{S} \cong S \otimes_R \widehat{R}$$
   is again the Frobenius map on $\widehat{R}$. is