HOMEWORK #5 – MATH 538
FALL 2013

DUE MONDAY, OCTOBER 28TH

(1) Suppose that $M = M_1 + M_2$. Is it true that $\mathrm{Ass}_R M = \mathrm{Ass}_R M_1 \cup \mathrm{Ass}_R M_2$?

(2) Suppose that M is Noetherian and $\phi : M \longrightarrow M$ is a module homomorphism. Show that $\ker \phi^n \cap \mathrm{im}\, \phi^n = \{0\}$ for some $n \gg 0$.

Hint: The kernels are ascending.

(3) Suppose that R is a ring and $Q \subseteq R$ is a prime ideal. For each $n > 0$ we define $Q^{\{(n)\}}$ to be $f^{-1}(Q^n R_Q)$. Show that $Q^{\{(n)\}}$ is the unique Q-primary component in the primary decomposition of Q^n. It is called the nth symbolic power of Q. Roughly speaking, it is the functions that vanish to degree n along Q.

(4) Suppose that $R = k[x, y, z]/\langle xy - z^2 \rangle$ and set $Q = \langle x, z \rangle \subseteq R$. Compute $Q^{\{(n)\}}$ for all $n \geq 0$ (at least try a few n until you see the pattern).

(5) Let $I = \langle x, y \rangle \cap \langle x, z \rangle \cap \langle y, z \rangle$. Find a primary decomposition of I^2.

(6) Suppose that $I \subseteq R$ is an ideal. Then for any R-module M we define

$$\Gamma_I(M) = \{x \in M \mid I^n x = 0 \text{ for some } n > 0.\}$$

(a) Show that $\Gamma_I(M)$ is a submodule of M.

(b) Suppose R is Noetherian and that M is finitely generated. Show that $\Gamma_I(M) = M$ if and only if $\mathrm{Supp}\, M \subseteq V(I)$.

(c) Show that $\Gamma_I(\bullet)$ is left-exact. In other words if $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ is exact, then $0 \longrightarrow \Gamma_I(L) \longrightarrow \Gamma_I(M) \longrightarrow \Gamma_I(N)$ is also exact.

(d) If L is injective and $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ is exact, show that $0 \longrightarrow \Gamma_I(L) \longrightarrow \Gamma_I(M) \longrightarrow \Gamma_I(N) \longrightarrow 0$ is exact.

(7) Suppose R is Noetherian. For any ideal $I = \langle f_1, \ldots, f_n \rangle \subset R$ with $U = (\mathrm{Spec}\, R) \setminus V(I)$.

(a) show first that $U = \bigcup_{i=1}^n \Big((\mathrm{Spec}\, R) \setminus V(f) \Big)$.

For simplicity we use R_{f_i} to denote $\{1, f_i, f_i^2, \ldots,\}^{-1} R$. Now, for any R-module M, we set $M_{f_i} = M \otimes_R R_{f_i}$ and we define $\Gamma(U, M)$ to be the kernel of the map

$$\bigoplus_{i=1}^{n} M_{f_i} \longrightarrow \bigoplus_{1 \leq i < j \leq n} M_{f_i f_j}$$

where the map sends $(\ldots, x_i/f_i^{m_i}, \ldots, x_j/f_j^{m_j}, \ldots)$ to $(\ldots, x_i/f_i^{m_i} - x_j/f_j^{m_j}, \ldots)$ (technically the sign choices should be slightly more complicated but it doesn't matter). There is a canonical map $M \longrightarrow \Gamma(U, M)$ which sends $x \mapsto (x/1, x/1, \ldots, x/1)$.

Show that the kernel of the canonical map $M \longrightarrow \Gamma(U, M)$ is exactly $\Gamma_I(M)$.