HOMEWORK #4 – MATH 538
FALL 2013

DUE FRIDAY, OCTOBER 18TH

(1) Let R be an integral domain and $K(R)$ its field of fractions. We define the *normalization* of R to be the integral closure of R in $K(R)$, in this class we will denote it by R^{N}. We say that R is *normal* if it is its own normalization.
  (a) Compute the normalization of $k[t]$, $k[t^2, t^3]$, $k[t(t-1), t^2(t-1)]$.
  (b) Show that the formation of R^{N} commutes with localization, in other words that $W^{-1}(R^{\mathrm{N}}) = (W^{-1}R)^{\mathrm{N}}$.
  (c) Suppose that $R \longrightarrow S$ is a map of rings. Do we always have an induced map between the normalizations? Find a sufficient condition which implies that there is such a map.

(2) Finish the proof of the going up theorem from the worksheet. Here is what you need to do.
  (a) Suppose that $R \subseteq S$ is integral Suppose that $Q \subseteq Q'$ are prime ideals of S and that $P = Q \cap R = Q' \cap R$. Show that $Q = Q'$.
  (b) [The going up theorem] Suppose that $R \subseteq S$ is integral. If $P_1 \subseteq \ldots \subseteq P_n$ is a chain of prime ideals of R and $Q_1 \subseteq \ldots \subseteq Q_m$ is a chain of prime ideals of R with $m < n$ and $Q_i \cap R = P_i$ for $1 \leq i \leq m$, then there exist $Q_{i+1} \subseteq \ldots \subseteq Q_n$, containing Q_i so that $Q_i \cap R = P_i$ for $1 \leq i \leq n$.

(3) Suppose $A \subseteq B$ is a ring extension and that $y, z \in B$ satisfy quadratic integral dependence relations $y^2 + ay + b = 0$ and $z^2 + cz + d = 0$ over A. Find explicit integral dependence relations for $y + z$ and yz.

(4) Recall that a topological space Y is called *irreducible* if whenever $Y = W \cup Z$ is decomposed as a union of closed sets, then $W = Y$ or $Z = Y$.
  (a) Suppose that $I \subseteq k[x_1, \ldots, x_n]$ and $k = \overline{k}$. Show that $\mathbb{V}(I) \subseteq k^n$ is irreducible if and only if $\sqrt{I}$ is prime.
  (b) Suppose that $I \subseteq R$ is a ring. Show that $V(I) \subseteq \operatorname{Spec} R$ is irreducible if and only if $\sqrt{I}$ is prime.

(5) Suppose that R is a Noetherian ring. Is it true that there exists an integer $n_0 > 0$ such that
  (a) every ideal $I \subseteq R$ is generated by at most n_0 elements?
  (b) every ascending chain of ideals $I_1 \subsetneq I_2 \subsetneq \ldots$ has length at most n_0?

  Prove or give a counter example.

(6) Let $A = k[x, y]$, $I = \langle x \rangle$ and $B = k$. We have the natural projection $f : A \longrightarrow A/I$ and the natural inclusion $g : k \hookrightarrow A/I \cong k[y]$. Consider the ring $C = \{(a, b) \in A \oplus B \mid f(a) = g(b)\}$ as in problem (9) from homework 2. Show that C is non-Noetherian even though it is a subring of a Noetherian ring.

(7) Show that every finitely generated module over a Noetherian ring is finitely presented.

(8) Show that every Artinian integral domain is a field.

(9) Suppose that $(R, \mathfrak{m})$ is an Artinian local ring. Show that $\mathfrak{m}$ is nilpotent, ie that $\mathfrak{m}^e = 0$ for $e \gg 0$.