HOMEWORK #3 – MATH 538
FALL 2013

DUE FRIDAY, OCTOBER 4TH

(1) Suppose that R is a ring and M and M' are R-modules. Prove that $M \oplus M'$ is flat if and only if M and M' are individually flat.

Solution: If $A \longrightarrow B$ is injective. Then we have a commutative diagram

$$\begin{tikzcd} A \otimes (M \oplus M') \arrow[r] \arrow[d, "\sim"] & B \otimes (M \oplus M') \arrow[d, "\sim"] \\ (A \otimes M) \oplus (A \otimes M') \arrow[r] & (B \otimes M) \oplus (B \otimes M') \end{tikzcd}$$

it is easy to see that the vertical maps are isomorphisms. Then we see immediately that the top horizontal map is injective (that is, $M \oplus M'$ is flat) if and only if the bottom horizontal map is injective. But the bottom is the direct sum of two maps, and that is injective if and only if each $A \otimes M \longrightarrow B \otimes M$ and $A \otimes M' \longrightarrow B \otimes M'$ are injective. In other words if M, and M' are flat.

(2) A R-module P is called *projective* if for every *surjective* map of R-modules $f : A \longrightarrow B$ and every R-module map $g : P \longrightarrow B$, there exists a map $h : P \longrightarrow A$ such that the following diagram commutes:

$$\begin{tikzcd} & P \arrow[dl, dotted, "h"'] \arrow[d, "g"] \\ A \arrow[r, "f"'] & B. \end{tikzcd}$$

(a) Show that a free module is projective.

Solution: Consider R^Γ a free module and suppose we are given $f : A \longrightarrow B$ and $g : R^\Gamma \longrightarrow B$. For each standard basis element $e_\gamma \in R^\Gamma$, consider $b_\gamma = g(e_\gamma)$. Choose a_γ such that $f(a_\gamma) = b_\gamma$. Finally we define $h(e_\gamma) = a_\gamma$. The diagram obviously commutes.

(b) Suppose that $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} P \longrightarrow 0$ is exact, prove that

$$0 \longrightarrow \operatorname{Hom}_R(P, M) \longrightarrow \operatorname{Hom}_R(B, M) \longrightarrow \operatorname{Hom}_R(A, M) \longrightarrow 0$$

is exact for every R-module M if P is projective.

Solution: (Sketch) We only need to show that $\operatorname{Hom}_R(B, M) \longrightarrow \operatorname{Hom}_R(A, M)$ surjects since the rest holds for any R-module P. Since P is projective, there exists a map $h : P \longrightarrow B$ such that $P \xrightarrow{h} B \xrightarrow{g} P$ is an isomorphism (make g the identity in the diagram above). It follows easily that $B \cong A \oplus P$ where f becomes the canonical injection and g becomes the canonical projection. To see this claim, consider map $A \oplus P \longrightarrow B$ which sends $(a, p) \mapsto f(a) + h(p)$. Given $b \in B$, we have $g(b) \in P$ and $h(g(b)) \in B$. Note $b - h(g(b))$ is sent to zero by g, so $b - h(g(b)) = f(a)$. Then $b = h(g(b)) + f(a)$ and setting $p = g(b)$ shows that $A \oplus P \longrightarrow B$ is surjective. To see it is injective, suppose that $(a, p) \mapsto f(a) + h(p) = 0$. Then $f(a) = h(-p)$ and so $0 = g(f(a)) = g(h(-p)) = -p$. But

once $-p = 0$, then $f(a) = 0$ and so $a = 0$ since f is injective. We have just shown the desired injectivity which proves that $A \oplus P \longrightarrow B$ is an isomorphism. One can then check that f and g are the canonical projections.
Then $\mathrm{Hom}_R(B, M) \cong \mathrm{Hom}_R(A \oplus P, M)$ and the surjectivity we desired is clear. Indeed, given $\phi : A \longrightarrow M$, we have the induced map $A \oplus P \longrightarrow M$ which sends P to zero and acts on A by ϕ, and thus a map in $\mathrm{Hom}_R(B, M)$. It is not difficult to see that this map is sent to ϕ by $\mathrm{Hom}_R(B, M) \cong \mathrm{Hom}_R(A \oplus P, M) \longrightarrow \mathrm{Hom}_R(A, M)$.

(c) Show that the following converse to (b) holds. If whenever $0 \longrightarrow A \longrightarrow B \longrightarrow P \longrightarrow 0$ is exact then

$$0 \longrightarrow \mathrm{Hom}_R(P, M) \longrightarrow \mathrm{Hom}_R(B, M) \longrightarrow \mathrm{Hom}_R(A, M) \longrightarrow 0$$

is also exact for every M, show that P is projective.

Solution: Suppose P satisfies the condition of (c). Choose any surjective map $F \xrightarrow{g} P$ where F is free (for instance, choose generators of P and map a free module onto P sending the basis elements to those generators). Let A denote the kernel of g with $f : A \longrightarrow F$ the induced map. Set $M = A$. Then we have that $\mathrm{Hom}_R(F, A) \longrightarrow \mathrm{Hom}_R(A, A)$ is surjective. In particular, there exists $\phi \in \mathrm{Hom}_R(F, A)$ such that $\mathrm{id}_A = \phi \circ f : A \xrightarrow{f} F \xrightarrow{\phi} A$. By the argument above, $F = P \oplus A$ and so P is a summand of a free module. The fact that it is then projective follows immediately by the same sort of argument we did in **1.**

(3) A R-module I is called *injective* if for every *injective* map of R-modules $f : A \longrightarrow B$ and every R-module map $g : A \longrightarrow I$, there exists a map $h : B \longrightarrow I$ such that the following diagram commutes:

$$\begin{tikzcd} I & \\ A \arrow[u, "g"] \arrow[r, "f"'] & B. \arrow[ul, dotted, "h"'] \end{tikzcd}$$

(a) Show that $\mathbb{Q}$ is an injective $\mathbb{Z}$-module, as is $\mathbb{Q}/\mathbb{Z}$.

Solution: Suppose we are given A, B and $I = \mathbb{Q}$ and maps f and g as in the above diagram. We need to show that g can be extended to h. Consider the set S of all pairs (C, h_C) where $A \subseteq C \subseteq B$ and $h_C|_A = g$, in other words, extensions of g. We give this set a partial order where $(C, h_C) \leq (C', h_{C'})$ if $C \subseteq C'$ and $h_{C'}|_C = h_C$. It is easy to see that given any ascending chain $\{C_\gamma, h_{C_\gamma}\}$, that the union is also in S and hence by Zorn's lemma, there exists a maximal element (C, h_C) in S. We will show that $C = B$ which will complete the proof.
Suppose $C \neq B$ and choose $b \in B \setminus C$. Consider $C' = C + \langle b \rangle_{\mathbb{Z}} \subseteq B$. We will construct $h_{C'}$ with $(C, h_C) \leq (C', h_{C'})$. There are two cases. Either $nb \in C$ for some integer $0 \neq n \in \mathbb{Z}$ or not. We need to define $h_{C'}(b)$.

(i) If $nb \in C$ then we define $h_{C'}(b) = h_C(nb)/n$. We verify that the induced map $h_{C'}$ which sends $c + mb \mapsto h_C(c) + mh_C(nb)/n$ is well defined. In particular if n_1, n_2 are nonzero integers such that $n_1 b, n_2 b \in C$, then $n_1 n_2 b \in C$ as well. But now

$$h_C(n_1 b)/n_1 = n_2 h_C(n_1 b)/(n_1 n_2) = n_1 h_C(n_2 b)/(n_1 n_2) = h_C(n_2 b)/n_2$$

More generally, suppose that $c_1+m_1b = c_2+m_2b \in C'$. Then we see that $(m_1-m_2)b = c_2 - c_1 \in C$. Then

$$\begin{array}{rl} & h_{C'}(c_1+m_1b) \\ = & h_C(c_1)+m_1h_C(nb)/n \\ = & h_C(c_1)+m_1h_C(nb)/n+h_C(c_2-c_1)-h_C((m_1-m_2)b) \\ = & h_C(c_2)+m_1h_C(nb)/n-h_C(n(m_1-m_2)b)/n \\ = & h_C(c_2)+m_1h_C(nb)/n-m_1h_C(nb)/n+m_2h_C(nb)/n \\ = & h_C(c_2)++m_2h_C(nb)/n \\ = & h_{C'}(c_2+m_2b) \end{array}$$

We need to also show it is a homomorphism so consider

$$\begin{array}{rl} & h_{C'}((c_1+m_1b)+(c_2+m_2b)) \\ = & h_{C'}((c_1+c_2)+(m_1+m_2)b) \\ = & h_C(c_1+c_2)+(m_1+m_2)h_C(nb)/n \\ = & h_C(c_1)+m_1h_C(nb)/n+h_C(c_2)+m_2h_C(nb)/n \\ = & h_{C'}(c_1+m_1b)+h_{C'}(c_2+m_2b) \end{array}$$

The multiplicative property is even easier so we leave that to the reader. But now we have $(C', h_{C'}) \geq (C, h_C)$ and $C' \supsetneq C$, a contradiction to the maximality of (C, h_C). Hence $C = B$ and we can set $h = h_C$.

(ii) If $nb \notin C$ for any $0 \neq n \in \mathbb{Z}$, then we define $h_{C'}(b) = 0$ so that in general $h_{C'}(c+mb) = h_C(c)$.

We now need to handle $\mathbb{Q}/\mathbb{Z}$. But the proof is essentially the same. Indeed, the only place we used that $\mathbb{Q}$ was $\mathbb{Q}$ was observing that if $x \in \mathbb{Q}$ then so is x/n. This property also holds for $\mathbb{Q}/\mathbb{Z}$.

(b) Suppose that $0 \longrightarrow I \longrightarrow B \longrightarrow C \longrightarrow 0$ is exact, prove that

$$0 \longrightarrow \mathrm{Hom}_R(N, I) \longrightarrow \mathrm{Hom}_R(N, B) \longrightarrow \mathrm{Hom}_R(N, C) \longrightarrow 0$$

is exact for every R-module N if I is injective.

Solution: This is exactly the same as the proof of (b) above (just some arrows were reversed to protect the innocent).

(c) Show that the following converse to (b) holds. If whenever $0 \longrightarrow I \longrightarrow B \longrightarrow C \longrightarrow 0$ is exact then

$$0 \longrightarrow \mathrm{Hom}_R(N, I) \longrightarrow \mathrm{Hom}_R(N, B) \longrightarrow \mathrm{Hom}_R(N, C) \longrightarrow 0$$

is also exact for every N, show that I is projective. (This might be quite hard, I don't know a good way to do it without using something that we haven't shown yet).

Solution: If we know that every module embeds into an injective module, then this is the same as above (again reversing arrows).

(4) Suppose that R is a ring.

(a) If M is a projective R-module, show that there exists another module P such that $M \oplus P$ is isomorphic to a free module.

Solution: This is pretty easy. Choose F a free module with a surjection $F \longrightarrow M$. Now let P be the kernel so we have a short exact sequence $0 \longrightarrow P \longrightarrow F \longrightarrow M \longrightarrow 0$. In our work in **2(b)**, we showed that $F \cong M \oplus P$.

(b) Suppose now that $M \oplus N$ is a free module. Show that M and N are projective.

Solution: Choose a surjective map $f : A \longrightarrow B$ and a map $g : M \longrightarrow B$. We can extend this map to a map $g' : M \oplus N \longrightarrow B$ which sends the summand N to zero. The fact that $M \oplus N$ is free implies it is projective and so the map h' exists in the following diagram.

$$\begin{tikzcd} & M \oplus N \arrow[dl] \arrow[d, "g'"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Restricting g' to M yields the result.

(c) Suppose that $W \subseteq R$ is a multiplicative system. Suppose that M is a projective module. Show that $W^{-1}M$ is also projective.

Solution: Since M is projective, it is a summand of a free module by (a). This property is obviously preserved under localization.

(d) Suppose that M is a *finitely presented* module such that M_P is projective as an R_P-module for each $P \in \operatorname{Spec} R$ (here the subscript P means localization at P, in other words inverting $W = R \setminus P$). Show that M is projective.

Solution: Suppose that $A \longrightarrow B$ is surjective, the diagram above is easily seen to be equivalent to showing that the induced map $\rho : \operatorname{Hom}_R(M, A) \longrightarrow \operatorname{Hom}_R(M, B)$ is also surjective. As we have seen, ρ is surjective if and only if $\rho_P : \big(\operatorname{Hom}_R(M, A)\big)_P \longrightarrow \big(\operatorname{Hom}_R(M, B)\big)_P$ is surjective for every $P \in \operatorname{Spec} R$. But since M is finitely presented, Hom commutes with localization and so we can identify ρ_P with

$$\operatorname{Hom}_{R_P}(M_P, A_P) \longrightarrow \operatorname{Hom}_{R_P}(M_P, B_P)$$

It is straightforward to see that this map is the application of $\operatorname{Hom}_{R_P}(M_P, \underline{\quad})$ to the induced map $A_P \longrightarrow B_P$. The result follows immediately.

(5) Suppose that M is a finitely generated projective module over a local ring R. Show that M is isomorphic to a free module. (This is true without the finitely generated bit too, but harder).

Solution: We choose $g : R^n \longrightarrow M$ surjective where n is minimal. Of course, we also have a map $s : M \longrightarrow R^n$ such that $g \circ s : M \xrightarrow{s} R^n \xrightarrow{g} M$ is an isomorphism. Now set $\mathfrak{m}$ to be the maximal ideal of R. We notice that $g/\mathfrak{m} : R^n/(\mathfrak{m} \cdot R^n) \longrightarrow M/(\mathfrak{m} \cdot M)$ is an isomorphism, hence so is $s/\mathfrak{m} : M/(\mathfrak{m} \cdot M) \longrightarrow R^n/(\mathfrak{m} \cdot R^n)$. But this implies that s is surjective. On the other hand, certainly s is injective, hence an isomorphism.

(6) Suppose R is a local ring. If $R^n \cong R^m$, show that $n = m$. Then prove the same result for non-local rings.

Solution: Ok, this result actually isn't true if $R = 0$ is the zero ring, but otherwise... Consider an isomorphism $R^n \longrightarrow R^m$. Modding out by a maximal ideal $\mathfrak{m}$ we get an isomorphism

$$(R/\mathfrak{m})^n \cong R^n/(\mathfrak{m} \cdot R^n) \xrightarrow{\sim} R^m/(\mathfrak{m} \cdot R^m) \cong (R/\mathfrak{m})^m.$$

But $R/\mathfrak{m}$ is a field and the result follows. I'll let you fill in any remaining justifications.

(7) Suppose that R is a local ring and M and N are finitely generated R-modules. If $M \otimes_R N = 0$ show that either $M = 0$ or $N = 0$.

Solution: We have a canonical surjective map $M \otimes_R N \longrightarrow \big(M/(\mathfrak{m} \cdot M)\big) \otimes_R \big(N/(\mathfrak{m} \cdot N)\big)$. It is also easy to see that

$$\big(M/(\mathfrak{m} \cdot M)\big) \otimes_R \big(N/(\mathfrak{m} \cdot N)\big) \cong \big(M/(\mathfrak{m} \cdot M)\big) \otimes_{(R/\mathfrak{m})} \big(N/(\mathfrak{m} \cdot N)\big)$$

since anything in $\mathfrak{m}$ already acts as zero. Now suppose that $M \neq 0$ and $N \neq 0$. We see that $M/(\mathfrak{m} \cdot M) \neq 0$ and $N/(\mathfrak{m} \cdot N) \neq 0$ by Nakayama's lemma. But they are now non-zero $R/\mathfrak{m}$-vector spaces, so that the tensor product $\big(M/(\mathfrak{m} \cdot M)\big) \otimes_{(R/\mathfrak{m})} \big(N/(\mathfrak{m} \cdot N)\big)$ is nonzero. The result follows.

(8) Suppose that A is a ring and

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

is a short exact sequence of A-modules. Show that if L and N are finitely generated A-modules, then so is M. Use this to show that if $M_1, M_2 \subseteq M$ are submodules such that $M_1 + M_2$ and $M_1 \cap M_2$ are finitely generated, so are M_1 and M_2.

Solution: First suppose that $n_1, \ldots n_d$ and $l_1, \ldots, l_e$ are generators of L and N respectively. Choose $m_1, \ldots, m_d$ mapping to the n_i via g. We want to show that $\{f(l_j)\}_{1 \leq j \leq e} \cup \{m_i\}_{1 \leq i \leq d}$ generate M. Choose $x \in M$. Write $g(x) = \sum_{i=1}^d r_i n_i$ and so consider $x - \sum_{i=1}^d r_i m_i \in M$. This might not be zero, but g of it is zero so that $x - \sum_{i=1}^d r_i m_i = \sum_{j=1}^e r'_j f(l_j)$. Solving for x solves the problem.

(9) Prove the 5-lemma. In other words, suppose that

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r, "a"] \arrow[d, "\alpha_1"'] & M_2 \arrow[r, "b"] \arrow[d, "\alpha_2"'] & M_3 \arrow[r, "c"] \arrow[d, "\alpha_3"'] & M_4 \arrow[r, "d"] \arrow[d, "\alpha_4"'] & M_5 \arrow[r] \arrow[d, "\alpha_5"'] & 0 \\
0 \arrow[r] & N_1 \arrow[r, "a'"'] & N_2 \arrow[r, "b'"'] & N_3 \arrow[r, "c'"'] & N_4 \arrow[r, "d'"'] & N_5 \arrow[r] & 0
\end{tikzcd}$$

is a commutative diagram with exact sequences as rows.

(a) If we suppose that α_2 and α_4 are surjective, and α_5 is injective, show that α_3 is surjective.
(b) If we suppose that α_2 and α_4 are injective, and α_1 is surjective, show that α_3 is injective.

Solution: I'll only prove (a) as (b) is similar. Choose $x \in N_3$. Let $y = c'(x)$ be the image of x in N_4. Since α_4 is surjective, there exists $z \in M_4$ with $\alpha_4(z) = y$. Note $\alpha_5(d(z)) = d'(y) = d'(c'(x)) = 0$ by exactness of the bottom row. Since α_5 is injective, this implies that $d(z) = 0$ and so there exists $w \in M_3$ with $c(w) = z$ by the exactness of the top row. We don't know whether $\alpha_3(w)$ is equal to x. But notice that

$$c'(x - \alpha_3(w)) = c'(x) - c'(\alpha_3(w)) = y - \alpha_4(c(w)) = y - \alpha_4(z) = y - y = 0.$$

By exactness of the bottom row there exists $v \in N_2$ such that $b'(v) = x - \alpha_3(w)$. Since α_2 is surjective, there exists $t \in M_2$ such that $\alpha_2(t) = v$. Then we see that

$$x - \alpha_3(w) = b'(v) = b'(\alpha_2(t)) = \alpha_3(b(t)).$$

Solving for x yields

$$x = \alpha_3(b(t)) + \alpha_3(w) = \alpha_3(b(t) + w)$$

which proves that α_3 is surjective as desired.