HOMEWORK #3 – MATH 538
FALL 2013

DUE FRIDAY, OCTOBER 4TH

(1) Suppose that R is a ring and M and M' are R-modules. Prove that $M \oplus M'$ is flat if and only if M and M' are individually flat.

(2) A R-module P is called *projective* if for every *surjective* map of R-modules $f : A \longrightarrow B$ and every R-module map $g : P \longrightarrow B$, there exists a map $h : P \longrightarrow A$ such that the following diagram commutes:

$$\begin{tikzcd} & P \arrow[dl, dotted, "h"'] \arrow[d, "g"] \\ A \arrow[r, "f"'] & B. \end{tikzcd}$$

   (a) Show that a free module is projective.

   (b) Suppose that $0 \longrightarrow A \longrightarrow B \longrightarrow P \longrightarrow 0$ is exact, prove that

$$0 \longrightarrow \mathrm{Hom}_R(P, M) \longrightarrow \mathrm{Hom}_R(B, M) \longrightarrow \mathrm{Hom}_R(A, M) \longrightarrow 0$$

   is exact for every R-module M if P is projective.

   (c) Show that the following converse to (b) holds. If whenever $0 \longrightarrow A \longrightarrow B \longrightarrow P \longrightarrow 0$ is exact then

$$0 \longrightarrow \mathrm{Hom}_R(P, M) \longrightarrow \mathrm{Hom}_R(B, M) \longrightarrow \mathrm{Hom}_R(A, M) \longrightarrow 0$$

   is also exact for ever M, show that P is projective.

(3) A R-module I is called *injective* if for every *injective* map of R-modules $f : A \longrightarrow B$ and every R-module map $g : A \longrightarrow I$, there exists a map $h : B \longrightarrow I$ such that the following diagram commutes:

$$\begin{tikzcd} I & \\ A \arrow[u, "g"] \arrow[r, "f"'] & B. \arrow[ul, dotted, "h"'] \end{tikzcd}$$

   (a) Show that $\mathbb{Q}$ is an injective $\mathbb{Z}$-module, as is $\mathbb{Q}/\mathbb{Z}$.

   (b) Suppose that $0 \longrightarrow I \longrightarrow B \longrightarrow C \longrightarrow 0$ is exact, prove that

$$0 \longrightarrow \mathrm{Hom}_R(N, I) \longrightarrow \mathrm{Hom}_R(N, B) \longrightarrow \mathrm{Hom}_R(N, C) \longrightarrow 0$$

   is exact for every R-module N if I is injective.

   (c) Show that the following converse to (b) holds. If whenever $0 \longrightarrow I \longrightarrow B \longrightarrow C \longrightarrow 0$ is exact then

$$0 \longrightarrow \mathrm{Hom}_R(N, I) \longrightarrow \mathrm{Hom}_R(N, B) \longrightarrow \mathrm{Hom}_R(N, C) \longrightarrow 0$$

   is also exact for every N, show that I is projective. (This might be quite hard, I don't know a good way to do it without using something that we haven't shown yet).

(4) Suppose that R is a ring.

   (a) If M is a projective R-module, show that there exists another module P such that $M \oplus P$ is isomorphic to a free module.

(b) Suppose now that $M \oplus N$ is a free module. Show that M and N are projective.
(c) Suppose that $W \subseteq R$ is a multiplicative system. Suppose that M is a projective module. Show that $W^{-1}M$ is also projective.
(d) Suppose that M is a *finitely presented* module such that M_P is projective as an R_P-module for each $P \in \operatorname{Spec} R$ (here the subscript P means localization at P, in other words inverting $W = R \setminus P$). Show that M is projective.

(5) Suppose that M is a finitely generated projective module over a local ring R. Show that M is isomorphic to a free module. (This is true without the finitely generated bit too, but harder).
(6) Suppose R is a local ring. If $R^n \cong R^m$, show that $n = m$. Then prove the same result for non-local rings.
(7) Suppose that R is a local ring and M and N are finitely generated R-modules. If $M \otimes_R N = 0$ show that either $M = 0$ or $N = 0$.
(8) Suppose that A is a ring and

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

is a short exact sequence of A-modules. Show that if L and N are finitely generated A-modules, then so is M. Use this to show that if $M_1, M_2 \subseteq M$ are submodules such that $M_1 + M_2$ and $M_1 \cap M_2$ are finitely generated, so are M_1 and M_2.
(9) Prove the 5-lemma. In other words, suppose that

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & M_4 & \longrightarrow & M_5 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \downarrow{\scriptstyle \alpha_4} & & \downarrow{\scriptstyle \alpha_5} & & \\
0 & \longrightarrow & N_1 & \longrightarrow & N_2 & \longrightarrow & N_3 & \longrightarrow & N_4 & \longrightarrow & N_5 & \longrightarrow & 0
\end{array}$$

is a commutative diagram with exact sequences as rows.
(a) If we suppose that α_2 and α_4 are surjective, and α_5 is injective, show that α_3 is surjective.
(b) If we suppose that α_2 and α_4 are injective, and α_1 is surjective, show that α_3 is injective.