HOMEWORK #1 - MATH 538 FALL 2013

SOLUTION TO PROBLEM (G)

Exercise (g) Suppose that k is a field. Consider the following set

$$S := \{ f \in k[x] \mid f(0) = f(1) \}.$$

Show that S is a ring. Describe S as a quotient of a polynomial ring and describe its prime Spectrum. Additionally, describe the induced map on Specs from the inclusion of rings $S \hookrightarrow k[x]$ at least in the case that k is algebraically closed.

Solution. First we show that S is a ring. Suppose $f, g \in S$. Then (f + g)(0) = f(0) + g(0) = f(1) + g(1) = (f + g)(1) and so S is closed under addition. Multiplication is the same. Clearly S contains 0 and 1. Since S is a subset of an existing ring k[x], we see that S automatically satisfies the rest of the ring axioms (distributivity, additive and multiplicative associativity and commutativity, etc.) Hence S is a subring of k[x].

Now we give a presentation of S. First consider the subset $H \subseteq S$ made up of elements f such that f(0) = f(1) = 0. This is clearly the ideal $\langle x(x-1) \rangle$ in the PID k[x], and so $H = \{g(x)x(x-1) \mid g(x) \in k[x]\}$. Of course, $S = \{h(x) + \lambda \mid h(x) \in H\}$. In particular, S is made up of the elements

$$g(x)x(x-1) + \lambda, g(x) \in k[x]$$

Since any $g(x) \in k[x]$ can be written as $\sum_{i=0}^{n} a_i x^i$, we see that

$$S = k[x(x-1), x \cdot x(x-1), x^2 \cdot x(x-1), x^3 \cdot x(x-1)] = k[x(x-1), x^2(x-1), x^3(x-1), \ldots]$$

$$\subseteq k[x].$$

On the other hand,

$$(x(x-1))^{2} + x^{2}(x-1) = (x^{3} - x^{2})(x-1) + x^{2}(x-1) = x^{3}(x-1)$$

so that $x^3(x-1)$ is not needed as a generator of the ring. More generally

$$(x(x-1))(x^{i-1}(x-1)) + x^{i}(x-1) = (x^{i+1} - x^{i})(x-1) + x^{i}(x-1) = x^{i+1}(x-1)$$

and so $S = k[x(x-1), x^2(x-1)]$. Setting a = x(x-1) and $b = x^2(x-1)$ we see we have the relation

$$a^{3} - b^{2} + ab$$

$$= x^{3}(x-1)^{3} - x^{4}(x-1)^{2} + x^{3}(x-1)^{2}$$

$$= x^{3}(x-1)^{2}((x-1)-x) + x^{3}(x-1)^{2}$$

$$= x^{3}(x-1)^{2}(-1) + x^{3}(x-1)^{2} = 0$$

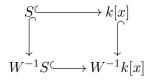
and so we have a surjective map $\phi : k[a,b]/\langle a^3 - b^2 + ab \rangle \to S$. We will see later that there are only 3 types of primes of k[a,b]. The zero ideal, principal primes, and maximal ideals. Taking this as a fact for now, since S is not a field, the kernel of $k[a,b] \to S$ can't be maximal. On the other hand, taking the fact that k[a,b] is a UFD on faith (you may have seen this before), we see that $\langle a^3 - b^2 + ab \rangle$ is prime and cannot be contained in any other principal prime ideal. It follows that ϕ is an isomorphism. There are certainly more direct (painful) ways to verify this, but I am unwilling to do that :-)

Finally, we need to understand the map π : Spec $k[x] \to$ Spec S. Certainly the ideal $Q = \langle x(x-1), x^2(x-1) \rangle_S \subseteq S$ is prime (in fact maximal since the quotient is k, a field). We can ask what ideals of k[x] have it as their image under π . Any such ideals certainly contain x(x-1), but there are only two such ideals $\langle x \rangle_{k[x]}$ and $\langle x - 1 \rangle_{k[x]}$. These two ideals must be sent to Q under π since Q is maximal and these are the only two ideals sent to Q. To understand the map on the rest of the Spec, we localize under the multiplicative system $W = \{x(x-1), x^2(x-1)^2, x^3(x-1)^3, \ldots\}$ which is a multiplicative system for both S and k[x].

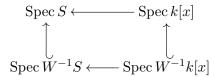
Note $W^{-1}k[x] = k[x, \frac{1}{x}, \frac{1}{x-1}]$ and

$$W^{-1}S = k[x(x-1), x^2(x-1), \frac{1}{x(x-1)}] = k[x(x-1), x^2(x-1), \frac{1}{x(x-1)}, \frac{x^2(x-1)}{x(x-1)}] = k[x, \frac{1}{x}, \frac{1}{x-1}] = W^{-1}k[x]$$

It follows that π is an isomorphism for the primes corresponding to primes of $W^{-1}S$ and $W^{-1}k[x]$. Indeed, to see this consider the following commutative diagram



and the induced diagram on Specs



But the only primes not of this form are those we already understood, $Q \subseteq S$ and $\langle x \rangle_{k[x]}$ and $\langle x - 1 \rangle_{k[x]}$ in k[x]. In particular, π is a bijection except over Q, where it is 2-to-1.