HOMEWORK #1 – MATH 538
FALL 2013

SOLUTION TO PROBLEM (G)

Exercise (g) Suppose that k is a field. Consider the following set

$$S := \{f \in k[x] \mid f(0) = f(1)\}.$$

Show that S is a ring. Describe S as a quotient of a polynomial ring and describe its prime Spectrum. Additionally, describe the induced map on Specs from the inclusion of rings $S \hookrightarrow k[x]$ at least in the case that k is algebraically closed.

Solution. First we show that S is a ring. Suppose $f, g \in S$. Then $(f+g)(0) = f(0) + g(0) = f(1) + g(1) = (f+g)(1)$ and so S is closed under addition. Multiplication is the same. Clearly S contains 0 and 1. Since S is a subset of an existing ring $k[x]$, we see that S automatically satisfies the rest of the ring axioms (distributivity, additive and multiplicative associativity and commutativity, etc.) Hence S is a subring of $k[x]$.

Now we give a presentation of S. First consider the subset $H \subseteq S$ made up of elements f such that $f(0) = f(1) = 0$. This is clearly the ideal $\langle x(x-1)\rangle$ in the PID $k[x]$, and so $H = \{g(x)x(x-1) \mid g(x) \in k[x]\}$. Of course, $S = \{h(x) + \lambda \mid h(x) \in H\}$. In particular, S is made up of the elements

$$g(x)x(x-1) + \lambda, g(x) \in k[x]$$

Since any $g(x) \in k[x]$ can be written as $\sum_{i=0}^{n} a_i x^i$, we see that

$$\begin{aligned} & S \\ = \; & k[x(x-1), x \cdot x(x-1), x^2 \cdot x(x-1), x^3 \cdot x(x-1)] \\ = \; & k[x(x-1), x^2(x-1), x^3(x-1), \dots] \\ \subseteq \; & k[x]. \end{aligned}$$

On the other hand,

$$\big(x(x-1)\big)^2 + x^2(x-1) = (x^3 - x^2)(x-1) + x^2(x-1) = x^3(x-1)$$

so that $x^3(x-1)$ is not needed as a generator of the ring. More generally

$$\big(x(x-1)\big)\big(x^{i-1}(x-1)\big) + x^i(x-1) = (x^{i+1} - x^i)(x-1) + x^i(x-1) = x^{i+1}(x-1)$$

and so $S = k[x(x-1), x^2(x-1)]$. Setting $a = x(x-1)$ and $b = x^2(x-1)$ we see we have the relation

$$\begin{aligned} & a^3 - b^2 + ab \\ = \; & x^3(x-1)^3 - x^4(x-1)^2 + x^3(x-1)^2 \\ = \; & x^3(x-1)^2((x-1) - x) + x^3(x-1)^2 \\ = \; & x^3(x-1)^2(-1) + x^3(x-1)^2 = 0 \end{aligned}$$

and so we have a surjective map $\phi : k[a,b]/\langle a^3 - b^2 + ab\rangle \longrightarrow S$. We will see later that there are only 3 types of primes of $k[a,b]$. The zero ideal, principal primes, and maximal ideals. Taking this as a fact for now, since S is not a field, the kernel of $k[a,b] \longrightarrow S$ can't be maximal. On the other hand, taking the fact that $k[a,b]$ is a UFD on faith (you may have seen this before), we see that $\langle a^3 - b^2 + ab\rangle$ is prime and cannot be contained in any other principal prime ideal. It follows that ϕ

is an isomorphism. There are certainly more direct (painful) ways to verify this, but I am unwilling to do that :-)

Finally, we need to understand the map $\pi : \operatorname{Spec} k[x] \longrightarrow \operatorname{Spec} S$. Certainly the ideal $Q = \langle x(x-1), x^2(x-1)\rangle_S \subseteq S$ is prime (in fact maximal since the quotient is k, a field). We can ask what ideals of $k[x]$ have it as their image under π. Any such ideals certainly contain $x(x-1)$, but there are only two such ideals $\langle x\rangle_{k[x]}$ and $\langle x-1\rangle_{k[x]}$. These two ideals must be sent to Q under π since Q is maximal and these are the only two ideals sent to Q. To understand the map on the rest of the Spec, we localize under the multiplicative system $W = \{x(x-1), x^2(x-1)^2, x^3(x-1)^3, \ldots\}$ which is a multiplicative system for both S and $k[x]$.

Note $W^{-1}k[x] = k[x, \frac{1}{x}, \frac{1}{x-1}]$ and

$$\begin{array}{rl} & W^{-1}S \\ = & k[x(x-1), x^2(x-1), \frac{1}{x(x-1)}] \\ = & k[x(x-1), x^2(x-1), \frac{1}{x(x-1)}, \frac{x^2(x-1)}{x(x-1)}] \\ = & k[x, \frac{1}{x}, \frac{1}{x-1}] \\ = & W^{-1}k[x] \end{array}$$

It follows that π is an isomorphism for the primes corresponding to primes of of $W^{-1}S$ and $W^{-1}k[x]$. Indeed, to see this consider the following commutative diagram

$$\begin{array}{ccc} S & \hookrightarrow & k[x] \\ \downarrow & & \downarrow \\ W^{-1}S & \hookrightarrow & W^{-1}k[x] \end{array}$$

and the induced diagram on Specs

$$\begin{array}{ccc} \operatorname{Spec} S & \longleftarrow & \operatorname{Spec} k[x] \\ \uparrow & & \uparrow \\ \operatorname{Spec} W^{-1}S & \longleftarrow & \operatorname{Spec} W^{-1}k[x] \end{array}$$

But the only primes not of this form are those we already understood, $Q \subseteq S$ and $\langle x\rangle_{k[x]}$ and $\langle x-1\rangle_{k[x]}$ in $k[x]$. In particular, π is a bijection except over Q, where it is 2-to-1.

$\square$