HOMEWORK #1 – MATH 538
FALL 2013

DUE WEDNESDAY, SEPTEMBER 4TH

(a) If I, J, J' are ideals, prove that $I \subseteq J \cup J'$ implies that either $I \subseteq J$ or $I \subseteq J'$. If P is a prime and $I \subseteq J \cup J' \cup P$, then show that I is contained in one of J, J' or P. This fact is a variant of a result called *prime avoidance*, and it will come up a couple times throughout the course.

(b) Suppose that I, J, K are ideals of R. Prove or give a counter-example to the following assertions:
   - $I \cap (J + K) = (I \cap J) + (I \cap K)$
   - If I and J are prime, so is $I + J$.
   - If $I + J = R$ then $IJ = I \cap J$.

(c) A ring R is called *reduced* if for any $x \in R$, $x^n = 0$ implies that $x = 0$. Find an example of a reduced ring that is not an integral domain. Justify your assertion.

(d) Suppose that R is reduced. Show that $\{0\} = \bigcap_{P \in \operatorname{Spec} R} P$.

(e) Suppose that R is a reduced ring that has finitely many minimal (with respect to inclusion) prime ideals $P_1, \ldots, P_n$. Show that the canonical map $R \longrightarrow \oplus_i^n R/P_i$ is an injection.

(f) Characterize all rings R such that $\operatorname{Spec} R = \emptyset$.

(g) Suppose that k is a field. Consider the following set

$$S := \{f \in k[x] \mid f(0) = f(1)\}.$$

Show that S is a ring. Describe S as a quotient of a polynomial ring and describe its prime Spectrum. Additionally, describe the induced map on Specs from the inclusion of rings $S \hookrightarrow k[x]$ at least in the case that k is algebraically closed.

(h) Suppose that R is an integral domain and $0 \neq r \in R$. Set $W = \{1, r, r^2, r^3, \ldots\}$. Show that $R[r^{-1}] \cong W^{-1}R \cong R[x]/\langle xr - 1\rangle$ where x is an indeterminant.