WORKSHEET # 1

MATH 538 FALL 2011

In this worksheet, we'll go through the proof of primary decomposition of ideals in a Noetherian ring. In particular, we will prove the following

Theorem. *In a Noetherian ring A, every ideal I has a primary decomposition. Thus each I can be written as*

$$I = Q_1 \cap \cdots \cap Q_n$$

where each Q_i is P_i-primary (for some prime P_i). Furthermore, if one writes the decomposition such that $P_i \neq P_j$ for all $i \neq j$ and such that $I \subsetneq (\cap_{i \neq j} Q_i)$ for each j (which is possible by **2.** *below), then the set of P_i which appear coincide exactly with $Ass(A/I)$.*

Before proving this, we first warm up.

1. Prove that Q is primary if and only if the only zero divisors in A/Q are nilpotent.

Solution: Indeed, suppose first that Q is primary and that $\overline{xy} = 0 \in A/Q$ with neither term being zero. Thus $xy = yx \in Q$ with $x, y \notin Q$ so that x^n and y^n are in Q for $n \gg 0$. Thus $\overline{x}$ and $\overline{y}$ are nilpotent.

Conversely if $xy \in Q$ with $x \neq 0$, then $\overline{xy} = 0$ with $\overline{x} = 0$. Thus either $\overline{y} = 0$ or $\overline{y}^n = 0$ for $n \gg 0$ (since in the second case, $\overline{y}$ must be nilpotent). But then $y^n \in Q$ as desired.

2. Prove that if Q_1 and Q_2 are both P-primary (to the same prime) then $Q_1 \cap Q_2$ is also P-primary.

Solution: We need to prove two things, that $Q_1 \cap Q_2$ is primary and that its radical is P.

First suppose that $xy \in Q_1 \cap Q_2$ with $x \notin Q_1 \cap Q_2$. Without loss of generality suppose that $x \notin Q_1$. Since Q_1 is primary, $y^n \in Q_1$ for $n \gg 0$, thus $y \in P = \sqrt{Q_2}$ and so $y^n \in Q_2$ for $n \gg 0$ as well. This proves the first part.

For the second, suppose that $y \in P = \sqrt{Q_1} = \sqrt{Q_2}$, then $y^n \in Q_1$ and $y^n \in Q_2$ for $n \gg 0$, and so $y^n \in Q_1 \cap Q_2$ for $n \gg 0$ as desired.

For the existence proof, we employ a strategy using the following definition.

Definition. An ideal $I \subseteq A$ is called *indecomposable* if it cannot be written as an intersection of two strictly bigger ideals.

Our strategy for proof of the existence part of the theorem has two parts. First show that every ideal can be written as an intersection of finitely many indecomposable ideals and then show that every indecomposable ideal is primary.

3. Show that if A is Noetherian, then every ideal is an intersection of finitely many indecomposable ideals.
Hint: Suppose not. Choose a largest such ideal and derive a contradiction.

Solution: Suppose that S is the set of such ideals which cannot be written as an intersection of finitely many indecomposable ideals. Since A is Noetherian, S has a largest element. Choose I to be that element. Then there are two possibilities, either I is indecomposable or it is not.

In the former case, then I can be written as an intersection of finitely many indecomposables (namely just itself). Thus this first case is impossible. In the latter case, if I is decomposable then $I = \mathfrak{a} \cap \mathfrak{b}$ with $I \subsetneq \mathfrak{a}, \mathfrak{b}$. But then since I is a largest element of S, neither $\mathfrak{a}$ or $\mathfrak{b}$ are in S and so $\mathfrak{a} = I_1 \cap \cdots \cap I_n$ for I_i indecomposable and $\mathfrak{b} = J_1 \cap \ldots J_m$ for J_j indecomposable. Thus

$$I = I_1 \cap \cdots \cap I_n \cap J_1 \cap \cdots \cap J_m$$

and we have just proven that I is not in S. A contradiction.

4. We will prove the following CLAIM: If B is a Noetherian ring and $\{0\} \subseteq B$ is indecomposable, then $\{0\}$ is primary. Explain why it is sufficient to prove this claim.

Solution: This will prove the second part of our proof strategy for existence. Indeed, if $B = A/Q$, then if Q is indecomposable, clearly $\langle 0 \rangle \subseteq B$ is indecomposable. Thus by the claim it is primary, and so by problem **2.** Q is also primary.

5. Now prove the CLAIM.
Hint: Suppose that $xy = 0$. Consider the ascending chain $\operatorname{Ann}_B y \subseteq \operatorname{Ann}_B y^2 \subseteq \operatorname{Ann}_B y^3 \subseteq \ldots$. Use the fact that the chain stabilizes to prove that $\langle x \rangle \cap \langle y^n \rangle = 0$ for n sufficiently large (analyze an element in that intersection). This is the hardest step in the worksheet.

Solution: Choose n such that the chain above stabilizes at n and fix $a \in \langle x \rangle \cap \langle y^n \rangle$. Thus $a \in \operatorname{Ann}_B y$ and $a = ty^n$. Thus $0 = ay = ty^{n+1}$ so $t \in \operatorname{Ann}_B y^{n+1} = \operatorname{Ann}_B y^n$ So $a = ty^n = 0$. This proves that $\langle x \rangle \cap \langle y^n \rangle = 0$ for $n \gg 0$. Since $\{0\}$ is indecomposable, either $\langle x \rangle$ or $\langle y^n \rangle = 0$ and the proof is complete.

Now prove the uniqueness theorem we discussed.

6. Prove that $\operatorname{Ass}(A/I) \subseteq \{P_1, \ldots, P_n\}$ where the P_i are in the Theorem.
Hint: Consider the diagonal map $A/I \to \oplus_{i=1}^n A/Q_i$. Prove that that map is injective to prove the containment $\subseteq$.

Solution: First we prove a lemma.

Lemma 0.1. *Suppose that* $0 \to L \to M \to N \to 0$ *is a short exact sequence. Then* $\operatorname{Ass}(M) \subseteq \operatorname{Ass}(L) \cup \operatorname{Ass}(N)$.

Proof. Suppose that $P \in \operatorname{Ass}(M)$ so that A/P is isomorphic to some $G \subseteq M$. There are two possibilities. Either $G \cap \operatorname{Ass}(M) = 0$ or $G \cap \operatorname{Ass}(M) \neq 0$. In the former case, then clearly N has a submodule isomorphic to G and thus to A/P. Thus $P \in \operatorname{Ass}(N)$. In the latter case, choose a non-zero $y \in G \cap L \subseteq G \cong A/P$. Set $x \in A/P$ to be the element corresponding to y. Then $\operatorname{Ann}_A y = \operatorname{Ann}_A x = P$ and so $P \in \operatorname{Ass}(L)$. $\square$

Now, certainly the map $A/I \cong A/(\cap_i Q_i) \to \oplus_{i=1}^n A/Q_i$ is injective. Thus if $P \in \operatorname{Ass}(A/I)$, then clearly $P \in \operatorname{Ass}(\oplus_{i=1}^n A/Q_i)$ as well (consider the composition of injections $A/P \cong G \to A/I \to \oplus_{i=1}^n A/Q_i$. But clearly the lemma implies that $\operatorname{Ass}(\oplus_{i=1}^n A/Q_i) \subseteq \{P_1, \ldots, P_n\}$ since we can write short exact sequences of the form $0 \to A/Q_1 \to \oplus_{i=1}^n A/Q_i \to \oplus_{i=2}^n A/Q_i \to 0$.

7. Prove that $\operatorname{Ass}(A/I) \supseteq \{P_1, \ldots, P_n\}$ where the P_i are in the Theorem.
Hint: consider $M_j = (\cap_{i \neq j} Q_i/I) \subseteq A/I$. Explain why $M_j \neq 0$ and consider its image in the direct sum above.

Solution: As originally written, I left out a key ingredient (although an obvious one) from the uniqueness part of the main theorem above. In particular, I need to assume that $I \subsetneq (\cap_{i \neq j} Q_i)$ for each j. It immediately follows that $M_j \neq 0$ for each j and so we consider it's non-zero image N in $\oplus_{i=1}^n A/Q_i$. Now, N is zero in each entry except the jth entry. Thus $N \cong M_j$ is a non-zero submodule of A/Q_j which clearly has only one associated prime P_j. Thus $\{P_j\} = \operatorname{Ass}(M_j)$ also.